Review Problems for Exam 2

1. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - y + 2z = 0 \right\}$. Find a basis for W consisting of vectors that are mutually orthogonal.

2. Let $v_1, v_2, \ldots, v_k$ be nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal; that is $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Prove that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

3. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which maps the parallelogram spanned by
$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
to the parallelogram spanned by
$$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
   (a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.
   (b) Compute $\det(T)$.
   (c) Show that T is invertible and compute the inverse of T.
   (d) Does T have real eigenvalues? If so, compute them and their corresponding eigenspaces.

4. Find a value of d for which the matrix
$$A = \begin{pmatrix} 1 & -2 \\ 3 & d \end{pmatrix}$$
is not invertible.

   Show that, for that value of d, $\lambda = 0$ is an eigenvalue of A. Give the eigenspace corresponding to 0. What is the dimension of $E_A(0)$?

5. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n, n)$ to compute $\det(A^{-1})$, provided that A is invertible.

6. Let A and B be $n \times n$ matrices. Show that if AB is invertible, then so is A.

7. Let A be a 3×3 matrix satisfying $A^3 - 6A^2 - 2A + 12I = O$, where I is the 3×3 identity matrix and O is the 3×3 zero matrix.

   (a) Prove that A is invertible and given a formula for computing its inverse in terms of I, A and A^2.

   (b) Prove that if λ is an eigenvalue of A, then $\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0$. Deduce therefore that λ is one of 6, $\sqrt{2}$ or $-\sqrt{2}$.

8. Let u denote a unit vector in $\mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(v) = \langle u, v\rangle u \quad \text{for all } v \in \mathbb{R}^n,$$

   where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

   (a) Verify that f is linear.

   (b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

   (c) The Dimension Theorem for a linear transformations, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, states that

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

   Use the Dimension Theorem to compute $\dim(\mathcal{N}_f)$.

9. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Assume that λ is an eigenvalue of T. Show that λ^m, for any positive integer m, is an eigenvalue for T^m, where T^m is the m–fold composition of T: $T^m = T \circ T \circ \cdots \circ$ (m times).

10. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be nilpotent if $T^k = O$, the zero transformation, for some positive integer k. Show that, if $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a nilpotent linear transformation, the $\lambda = 0$ is the only eigenvalue of T.

11. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be involution if $T^2 = I$, the identity transformation in $\mathbb{R}^n$. Assume $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an involution. Show that, if λ is an eigenvalue of T, then either $\lambda = 1$ or $\lambda = -1$.

12. Let A denote an $n \times n$ matrix. Suppose that $AA^T = I$, the $n \times n$ identity matrix. Assume that λ an eigenvalue of A^T. Show that $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A.