Solutions to Review Problems for Exam 2

1. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - y + 2z = 0 \right\}$. Find a basis for W consisting of vectors that are mutually orthogonal.

***Solution**:* We first note that $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Set

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

We then have that $\{v_1, v_2\}$ is a basis for W and $\dim(W) = 2$.

Next, we look for a basis, $\{w_1, w_2\}$, of W made up of orthogonal vectors.

Set $w_1 = v_1$ and look for $w \in \text{span}(\{v_1, v_2\})$ with the property that

$$\langle w, v_1 \rangle = 0. \tag{1}$$

Write $w = c_1 v_1 + c_2 v_2$ and substitute into (1) to get

$$\langle c_1 v_1 + c_2 v_2, v_1 \rangle = 0,$$

or

$$c_1 \langle v_1, v_1 \rangle + c_2 \langle v_2, v_1 \rangle = 0, \tag{2}$$

where we have used the bi–linearity of the inner product.

Next, compute

$$\langle v_1, v_1 \rangle = 2 \quad \text{and} \quad \langle v_2, v_1 \rangle = -2$$

and substitute into (2) to get the equation

$$2c_1 - 2c_2 = 0,$$

or

$$c_1 - c_2 = 0. \tag{3}$$

The equation in (3) has infinitely many solutions given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4}$$

Taking $t = 1$ in (4) we get that $c_1 = c_2 = 1$, so that

$$w = v_1 + v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

is lies in W and is orthogonal to w_1. Set

$$w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Then, $\{w_1, w_2\}$ is a basis for W made up of orthogonal vectors. □

2. Let $v_1, v_2, \ldots, v_k$ be nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal; that is $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Prove that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof: Assume that $v_1, v_2, \ldots, v_k$ are nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal.

Suppose that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0. \tag{5}$$

Take inner product with v_1 on both sides of (5) to get

$$\langle c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, v_1 \rangle = \langle 0, v_1 \rangle. \tag{6}$$

Next, apply the bi–linearity of the inner product on the left–hand side of (6) to get

$$c_1 \langle v_1, v_1 \rangle + c_2 \langle v_2, v_1 \rangle + \cdots + c_k \langle v_k, v_1 \rangle = 0,$$

so that

$$c_1 \|v_1\|^2 = 0, \tag{7}$$

where we have used the orthogonality assumption.

It follows from (7) and the assumption that $v_1 \neq 0$ that $c_1 = 0$. Similarly, taking the inner product with v_j, for $j = 2, 3, \ldots, k$, on both sides of (5) yields that $c_j = 0$ for $j = 2, 3, \ldots, k$. We have therefore shown that (5) implies that

$$c_1 = c_2 = \cdots = c_k = 0.$$

Hence, the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. □

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which maps the parallelogram spanned by

$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

to the parallelogram spanned by

$$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.

***Solution**:* Assume that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T(v_1) = w_1$ and $T(v_2) = w_2$. We compute

$$M_T = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}. \tag{8}$$

In order to compute $T(e_1)$, first we write e_1 in terms of v_1 and v_2 so that

$$c_1 v_1 + c_2 v_2 = e_1, \tag{9}$$

or

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{10}$$

The system in (10) can be solved by multiplying on both sides (on the left) by

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix},$$

so that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}. \tag{11}$$

It follows from (9) and (11) that

$$e_1 = \frac{1}{4} v_1 + \frac{1}{4} v_2. \tag{12}$$

Applying T on both sides of (12) and using the linearity of T, we obtain that

$$\begin{aligned} T(e_1) &= \frac{1}{4} T(v_1) + \frac{1}{4} T(v_2) \\ &= \frac{1}{4} w_1 + \frac{1}{4} w_2; \end{aligned}$$

so that

$$T(e_1) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}. \tag{13}$$

Similar calculations lead to that

$$T(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{14}$$

Combining (8), (13) and (14), we obtain that the matrix representation, M_T, or T, relative to the standard basis in $\mathbb{R}^2$ is

$$M_T = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}.$$

□

(b) Compute $\det(T)$. Does T preserve orientation?

***Solution**:* Compute

$$\det(T) = \det(M_T) = -\frac{1}{2}.$$

□

(c) Show that T is invertible and compute the inverse of T.

***Solution**:* Since $\det(T) \neq 0$, T is invertible, and the matrix representation for the inverse of T is given by

$$M_T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} 0 & -1 \\ -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Consequently, the inverse of T is given by

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x \end{pmatrix}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. □

(d) Does T have real eigenvalues? If so, compute them and their corresponding eigenspaces.

***Solution**:* The eigenvalues of T are scalars, λ, for which the system of equations

$$(M_T - \lambda I)v = \mathbf{0} \tag{15}$$

has nontrivial solutions. The system in (15) has nontrivial solutions if and only if the determinant of the matrix

$$M_T - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 1/2 & -\lambda \end{pmatrix}$$

is zero; that is,

$$\det(M_T - \lambda I) = 0,$$

or

$$\lambda^2 - \frac{1}{2} = 0.$$

Thus, $\lambda_1 = -\dfrac{1}{\sqrt{2}}$ and $\lambda_2 = \dfrac{1}{\sqrt{2}}$ are eigenvalues of T.

To find the eigespace corresponding to λ_1 we solve the homogenous system in (15) for $\lambda = \lambda_1$. We can do this by performing row operations of the augmented matrix

$$\left(\begin{array}{ccc} \frac{1}{\sqrt{2}} & 1 & | & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & | & 0 \end{array}\right),$$

which is row–equivalent to the matrix

$$\left(\begin{array}{ccc} 1 & \sqrt{2} & | & 0 \\ 0 & 0 & | & 0 \end{array}\right).$$

Thus, the system in (15) for $\lambda = \lambda_1$ is equivalent to the homogeneous equation

$$x_1 + \sqrt{2}\, x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$

Thus, the eigenspace of T associated with $\lambda_1 = -\dfrac{1}{\sqrt{2}}$ is

$$E_T(\lambda_1) = \operatorname{span}\left\{\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}\right\}.$$

Similarly, we can compute the eigenspace of T associated with $\lambda_2 = \dfrac{1}{\sqrt{2}}$ to be

$$E_T(\lambda_2) = \operatorname{span}\left\{\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}\right\}.$$

□

4. Find a value of d for which the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 3 & d \end{pmatrix}$$

is not invertible.

Show that, for that value of d, $\lambda = 0$ is an eigenvalue of A. Give the eigenspace corresponding to 0. What is the dimension of $E_A(0)$?

***Solution**:* The matrix A fails to be invertible when $\det(A) = 0$. This occurs when $d = -6$. For this value of d, the matrix A becomes

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$$

and observe that its second column is a multiple of the first. Therefore, the columns of A are linearly dependent; hence, the system

$$Av = \mathbf{0} \tag{16}$$

has nontrivial solutions and therefore $\lambda = 0$ is an eigenvalue of A. To find the corresponding eigenspace, observe that the system in (16) is equivalent to the equation

$$x_1 - 2x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, the eigenspace of A associated with $\lambda = 0$ is

$$E_A(0) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore, $\dim(E_A(0)) = 1$. □

5. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n, n)$ to compute $\det(A^{-1})$, provided that A is invertible.

Proof: Assume that A is invertible with inverse A^{-1}. Then,

$$A^{-1}A = I,$$

where I is the $n \times n$ identity matrix. Taking determinants on both sides of the equation yields that

$$\det(A^{-1}A) = 1,$$

from which we get that

$$\det(A^{-1})\det(A) = 1.$$

This, since $\det(A) \neq 0$ because A is invertible, we get that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

□

6. Let A and B be $n \times n$ matrices. Show that if AB is invertible, then so is A.

   Proof: Suppose that AB is invertible. Then, there exists an $n \times n$ matrix, C, such that

$$(AB)C = I,$$

   where I is the $n \times n$ identity matrix. Thus, by associativity of matrix multiplication,

$$A(BC) = I \tag{17}$$

   Applying the determinant on both sides of (17) we obtain that

$$\det(A) \cdot \det(BC) = 1,$$

   from which we get that $\det(A) \neq 0$. Hence, A is invertible. □

7. Let A be a 3×3 matrix satisfying $A^3 - 6A^2 - 2A + 12I = O$, where I is the 3×3 identity matrix and O is the 3×3 zero matrix.

   (a) Prove that A is invertible and given a formula for computing its inverse in terms of I, A and A^2.

   ***Solution**:* We can solve the equation $A^3 - 6A^2 - 2A + 12I = O$ for $12I$ and then divide by 12 to get that

$$A\left(\frac{1}{6}I + \frac{1}{2}A - \frac{1}{12}A^2\right) = I,$$

   which shows that A has a right–inverse and is therefore invertible with

$$A^{-1} = \frac{1}{6}I + \frac{1}{2}A - \frac{1}{12}A^2.$$

□

(b) Prove that if λ is an eigenvalue of A, then $\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0$. Deduce therefore that λ is one of 6, $\sqrt{2}$ or $-\sqrt{2}$.

Proof: Let λ be an eigenvalue of A. Then, there exists a nonzero vector, v, in $\mathbb{R}^3$ such that

$$Av = \lambda v.$$

Multiplying on both sides by A we then get that

$$A^2 v = \lambda A v = \lambda(\lambda v) = \lambda^2 v.$$

Multiplying the last equation by A we then get that

$$A^3 v = \lambda^3 v.$$

Thus, applying $A^3 - 6A^2 - 2A + 12I = O$ to to v we get that

$$(A^3 - 6A^2 - 2A + 12I)v = Ov,$$

which, by the distributive property, implies that

$$A^3 v - 6A^2 v - 2Av + 12v = \mathbf{0}.$$

Thus,

$$\lambda^3 v - 6\lambda^2 v - 2\lambda v + 12v = \mathbf{0},$$

or

$$(\lambda^3 - 6\lambda^2 - 2\lambda + 12)v = \mathbf{0},$$

from which we get that

$$\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0,$$

since v is nonzero.

Observe that $\lambda^3 - 6\lambda^2 - 2\lambda + 12$ factors into $(\lambda - 6)(\lambda + \sqrt{2})(\lambda - \sqrt{2})$. □

8. Let u denote a unit vector in $\mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(v) = \langle u, v\rangle u \quad \text{for all } v \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

(a) Verify that f is linear.

***Solution**:* For $v, w \in \mathbb{R}^n$, compute

$$\begin{aligned} f(v+w) &= \langle u, v+w\rangle u \\ &= (\langle u, v\rangle + \langle u, w\rangle)u \\ &= \langle u, v\rangle u + \langle u, w\rangle u \\ &= f(v) + f(w). \end{aligned}$$

Similarly, for a scalar c and $v \in \mathbb{R}^n$,

$$\begin{aligned} f(cv) &= \langle u, cv\rangle u \\ &= c\langle u, v\rangle u \\ &= cf(v). \end{aligned}$$

□

(b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

***Solution**:* The image of f is the set

$$\mathcal{I}_f = \{w \in \mathbb{R}^n \mid w = f(v) \text{ for some } v \in \mathbb{R}^n\}.$$

We claim that $\mathcal{I}_f = \text{span}\{u\}$. To see why this is so, first observe that $f(u) = \langle u, u\rangle u = \|u\|^2 u = u$, since u is a unit vector. Thus,

$$f(u) = u. \tag{18}$$

Let $w \in \text{span}\{u\}$; then $w = cu$, for some scalar c. Now, by the linearity of f,

$$w = cu = cf(u) = f(cu),$$

where we have used (18). We have therefor shown that

$$w \in \text{span}\{u\} \Rightarrow w \in \mathcal{I}_f;$$

that is,

$$\text{span}\{u\} \subseteq \mathcal{I}_f. \tag{19}$$

Next, suppose that $w \in \mathcal{I}_f$; then, $w = f(v)$ for some $v \in \mathbb{R}^n$, so that

$$w = \langle u, v\rangle u \in \text{span}\{u\}.$$

Thus,

$$\mathcal{I}_f \subseteq \text{span}\{u\}. \tag{20}$$

Combining (19) and (20) yields that

$$\mathcal{I}_f = \text{span}\{u\}.$$

It then follows that

$$\dim(\mathcal{I}_f) = 1. \tag{21}$$

The null space of f is the set

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid f(v) = \mathbf{0}\}.$$

Thus,

$$\begin{aligned} v \in \mathcal{N}_f \quad & \text{iff} \quad \langle u, v\rangle u = \mathbf{0} \\ & \text{iff} \quad \langle u, v\rangle = 0, \end{aligned}$$

since $u \neq \mathbf{0}$. It then follows that

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\};$$

that is, $\mathcal{N}_f$ is the space of vectors which are orthogonal to u. □

(c) The Dimension Theorem for a linear transformations, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, states that

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

Use the Dimension Theorem to compute $\dim(\mathcal{N}_f)$.

***Solution**:* Using the dimension theorem and (21) we get that

$$\dim(\mathcal{N}_f) + 1 = n,$$

which implies that

$$\dim(\mathcal{N}_f) = n - 1.$$

□

9. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Assume that λ is an eigenvalue of T. Show that λ^m, for some positive integer m, is an eigenvalue for T^m, where T^m is the m–fold composition of T: $T^m = T \circ T \circ \cdots \circ$ (m times).

***Solution**:* Let λ denote an eigenvalue of the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. Then, there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$T(v) = \lambda v. \tag{22}$$

Applying T to both sides of (22) we obtain

$$T(T(v)) = T(\lambda v);$$

so that, using the linearity of T and the definition of the m–fold composition of T,

$$T^2(v) = \lambda T(v).$$

Thus, since v is an eigenvector of T corresponding to λ,

$$T^2(v) = \lambda(\lambda v),$$

or

$$T^2(v) = \lambda^2 v.$$

Hence, λ^2 is an eigenvalue for T^2.

We may now proceed by induction on m. Having shown that λ^{m-1} is an eigenvalue of T^{m-1}, we show that λ^m is an eigenvalue of T^m. Thus, assume that λ^{m-1} denote an eigenvalue of T^{m-1}. Then, there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$T^{m-1}(v) = \lambda^{m-1} v. \tag{23}$$

Applying T to both sides of (23) we obtain

$$T(T^{m-1}(v)) = T(\lambda^{m-1} v);$$

so that, using the linearity of T and the definition of the m–fold composition of T,

$$T^m(v) = \lambda^{m-1} T(v).$$

Thus, since v is an eigenvector of T corresponding to λ,

$$T^m(v) = \lambda^{m-1}(\lambda v),$$

or

$$T^m(v) = \lambda^m v.$$

Hence, λ^m is an eigenvalue for T^m. □

10. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be nilpotent if $T^k = O$, the zero transformation, for some positive integer k. Show that, if $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a nilpotent linear transformation, the $\lambda = 0$ is the only eigenvalue of T.

    ***Solution**:* Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by a linear transformation satisfying $T^k = O$ for some positive integer k. Let λ be an eigenvalue for T; then, by the result of

Problem 9, λ^k is an eigenvalue of T^k. Thus, there exists a nonzero vector, $v \in \mathbb{R}^n$, such that

$$T^k(v) = \lambda^k v. \tag{24}$$

Thus, since T^k is the zero transformation in $\mathbb{R}^n$, it follows from (24) that

$$\lambda^k v = \mathbf{0}. \tag{25}$$

Hence, since v is nonzero, we obtain from (25) that $\lambda^k = 0$, which implies that $\lambda = 0$. □

11. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be involution if $T^2 = I$, the identity transformation in $\mathbb{R}^n$. Assume $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an involution. Show that, if λ is an eigenvalue of T, then either $\lambda = 1$ or $\lambda = -1$.

    ***Solution**:* Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by a linear transformation satisfying $T^2 = I$ and let λ be an eigenvalue for T; then, by the result of Problem 9, λ^2 is an eigenvalue of T^2. Thus, there exists a nonzero vector, $v \in \mathbb{R}^n$, such that

    $$T^2(v) = \lambda^2 v. \tag{26}$$

    Thus, since $T^2 = I$, the identity transformation in $\mathbb{R}^n$, it follows from (26) that

    $$\lambda^2 v = v,$$

    or

    $$(\lambda^2 - 1)v = \mathbf{0}. \tag{27}$$

    Hence, since v is nonzero, we obtain from (27) that $\lambda^2 = 1$, which implies that either $\lambda = -1$ or $\lambda = 1$. □

12. Let A denote an $n \times n$ matrix. Suppose that $AA^T = I$, the $n \times n$ identity matrix. Assume that λ an eigenvalue of A^T. Show that $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A.

    ***Solution**:* Assume that

    $$AA^T = I \tag{28}$$

    and that λ an eigenvalue of A^T.

    First we see that λ cannot be 0. Take the determinant on both sides of (28) to get

    $$\det(AA^T) = \det(I) = 1,$$

or

$$\det(A)\det(A^T) = 1,$$

from which we get that $\det(A^T) \neq 0$. Hence, the columns of A^T are linearly independent, and therefore the equation

$$A^T v = \mathbf{0}$$

has only the trivial solution. Consequently, 0 cannot be an eigenvalue of A^T. Hence, $\lambda \neq 0$.

There exists a nonzero vector, v, in $\mathbb{R}^n$ such that

$$A^T v = \lambda v. \tag{29}$$

Multiply on both sided of the equation in (29) by the matrix A on the left to get

$$A(A^T v) = A(\lambda v);$$

so that, by the associative property of matrix multiplication,

$$(AA^T)v = \lambda A v,$$

or

$$\lambda A v = v, \tag{30}$$

since $AA^T = I$. It follows from (30) and the fact that $\lambda \neq 0$ that

$$Av = \frac{1}{\lambda} v,$$

and therefore $1/\lambda$ is an eigenvalue of A. $\square$