## Solutions to Exam 2

1. Complete the following definitions:

   (a) A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear iff ...

   ***Answer:*** A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear iff for any vectors $v$ and $w$ in $\mathbb{R}^n$, and for any scalar $c$,

   (i) $f(v + w) = f(v) + f(w)$, and

   (ii) $f(cv) = cf(v)$.

   $\square$

   (b) A scalar, $\lambda$, is an eigenvalue of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ iff ...

   ***Answer:*** A scalar, $\lambda$, is an eigenvalue of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ iff the equation

   $$T(v) = \lambda v,$$

   has a nontrivial solution in $\mathbb{R}^n$. $\square$

   (c) The null space, $\mathcal{N}_T$, of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined to be ...

   ***Answer:*** The null space, $\mathcal{N}_T$, of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined to be the set

   $$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\}.$$

   $\square$

2. Let $\mathcal{B} = \{v_1, v_2\}$ be made up of the vectors

   $$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

   let $\mathcal{B}' = \{w_1, w_2\}$ be made up of the vectors

   $$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

   and $\mathcal{E} = \{e_1, e_2\}$ be the standard basis in $\mathbb{R}^2$.

   Let $id\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity map in $\mathbb{R}^2$.

(a) Compute the change of basis matrices $[id]_{\mathcal{B}}^{\mathcal{E}}$ and $[id]_{\mathcal{B}'}^{\mathcal{E}}$.

***Solution**:* We first compute

$$\begin{aligned}[id]_{\mathcal{B}}^{\mathcal{E}} &= \begin{bmatrix}[id(v_1)]_{\mathcal{E}} & [id(v_2)]_{\mathcal{E}}\end{bmatrix} \\ &= \begin{bmatrix}[v_1]_{\mathcal{E}} & [v_2]_{\mathcal{E}}\end{bmatrix} \\ &= \begin{bmatrix}v_1 & v_2\end{bmatrix};\end{aligned}$$

so that

$$[id]_{\mathcal{B}}^{\mathcal{E}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1}$$

Similarly,

$$[id]_{\mathcal{B}'}^{\mathcal{E}} = \begin{bmatrix}w_1 & w_2\end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2}$$

□

(b) Use your results from part (a) to compute the change of basis matrices $[id]_{\mathcal{E}}^{\mathcal{B}}$ and $[id]_{\mathcal{E}}^{\mathcal{B}'}$.

***Solution**:* The matrices $[id]_{\mathcal{E}}^{\mathcal{B}}$ and $[id]_{\mathcal{E}}^{\mathcal{B}'}$ are the inverses of the matrices in (1) and (2), respectively. Thus,

$$[id]_{\mathcal{E}}^{\mathcal{B}} = \left([id]_{\mathcal{B}}^{\mathcal{E}}\right)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

and

$$[id]_{\mathcal{E}}^{\mathcal{B}'} = \left([id]_{\mathcal{B}'}^{\mathcal{E}}\right)^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

or

$$[id]_{\mathcal{E}}^{\mathcal{B}'} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

□

(c) Use your results from parts (a) and (b) to compute the change of basis matrix $[id]_{\mathcal{B}}^{\mathcal{B}'}$.

***Solution**:* Compute

$$[id]_{\mathcal{B}}^{\mathcal{B}'} = [id]_{\mathcal{E}}^{\mathcal{B}'}[id]_{\mathcal{B}}^{\mathcal{E}},$$

where the matrices $[id]_{\mathcal{E}}^{\mathcal{B}'}$ and $[id]_{\mathcal{B}}^{\mathcal{E}}$ are given in (3) and (1), respectively; so that,

$$[id]_{\mathcal{B}}^{\mathcal{B}'} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}.$$

□

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T(e_1) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad T(e_3) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

where $\{e_1, e_2, e_3\}$ is the standard basis in $\mathbb{R}^3$.

(a) Give the matrix representation of $T$ relative to the standard basis in $\mathbb{R}^3$.
***Solution***:

$$M_{_T} = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

□

(b) Given that $\lambda = 1$ is an eivenvalue of the transformation $T$, compute the eigenspace, $E_T(1)$, corresponding to this eigenvalue. What is $\dim(E_T(1))$?
***Solution***: Solve the equation

$$(M_{_T} - I)v = \mathbf{0}, \tag{4}$$

by reducing the augmented matrix

$$\begin{pmatrix} -1 & 1 & -1 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ -1 & 1 & -1 & | & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix};$$

so that the system in (4) is equivalent to the equation

$$x_1 - x_2 + x_3 = 0. \tag{5}$$

Solving the equation in (5) for the leading variable $x_1$ leads to

$$x_1 = x_2 - x_3;$$

and setting $x_2 = t$ and $x_3 = -s$, where $t$ and $s$ are arbitrary parameters leads to

$$\begin{aligned} x_1 &= t + s \\ x_2 &= t \\ x_3 &= -s; \end{aligned}$$

so that, the solution space to the system in (4) is the set

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad t, s \in \mathbb{R} \right\}.$$

It then follows that the eigenspace corresponding to the eigenvalue $\lambda = 1$ is

$$E_T(1) = \text{span}\left( \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \right).$$

Hence, $\dim(E_T(1)) = 2$. □

4. Let $u_1$ and $u_2$ denote a unit vector in $\mathbb{R}^3$ that are orthogonal to each other; i.e., $\langle u_1, u_2 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$.

   (a) Define $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $f(v) = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$ for all $v \in \mathbb{R}^3$. Verify that $f$ is linear.

   ***Solution**:* Let $v$ and $w$ be vectors in $\mathbb{R}^3$ and compute

$$f(v + w) = \langle v + w, u_1 \rangle u_1 + \langle v + w, u_2 \rangle u_2;$$

   so that, using the bi–linearity of the Euclidean inner product and the distributive and associative properties for real numbers,

$$\begin{aligned} f(v + w) &= (\langle v, u_1 \rangle + \langle w, u_1 \rangle) u_1 + (\langle v, u_2 \rangle + \langle w, u_2 \rangle) u_2 \\ &= \langle v, u_1 \rangle u_1 + \langle w, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle w, u_2 \rangle u_2 \\ &= \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle w, u_1 \rangle u_1 + \langle w, u_2 \rangle u_2 \\ &= f(v) + f(w). \end{aligned}$$

   Similarly, for $v \in \mathbb{R}^3$ and $c \in \mathbb{R}$,

$$f(cv) = \langle cv, u_1 \rangle u_1 + \langle cv, u_2 \rangle u_2;$$

   so that, using the linearity of the Euclidean inner product □

(b) Verify that the set $\mathcal{B} = \{u_1, u_2\}$ is a basis for the image, $\mathcal{I}_f$, of $f$.

***Solution**:* First, we show that

$$\mathcal{I}_f = \text{span}(\{u_1, u_2\}). \tag{6}$$

Let $w \in \mathcal{I}_f$; then, there exists $v \in \mathbb{R}^3$ such that

$$w = f(v) = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2,$$

which is a linear combination of $u_1$ and $u_2$. We have therefore shown that

$$\mathcal{I}_f \subseteq \text{span}(\{u_1, u_2\}). \tag{7}$$

In order to show the other inclusion, note that

$$f(u_1) = \langle u_1, u_1\rangle u_1 + \langle u_1, u_2\rangle u_2 = u_1,$$

since $\langle u_1, u_2\rangle = 0$ and $u_1$ is a unit vector. Thus, $u_1 = f(u_1)$; so that $u_1 \in \mathcal{I}_f$. Similarly, $u_2 \in \mathcal{I}_f$. We then have that

$$\{u_1, u_2\} \subseteq \mathcal{I}_f,$$

from which we get that

$$\text{span}(\{u_1, u_2\}) \subseteq \mathcal{I}_f, \tag{8}$$

since $\mathcal{I}_f$ is a subspace of $\mathbb{R}^3$ and $\text{span}(\{u_1, u_2\})$ is the smallest subspace of $\mathbb{R}^3$ that contains $\{u_1, u_2\}$. Combining (7) and (8) yields (6).

Next, we show that $\{u_1, u_2\}$ is linearly independent.

Consider the equation

$$c_1u_1 + c_2u_2 = \mathbf{0}. \tag{9}$$

Take the inner product with $u_1$ on both sides of (9) to get

$$\langle c_1u_1 + c_2u_2, u_1\rangle = \langle \mathbf{0}, u_1\rangle,$$

or, using the bi–linearity of the inner product,

$$c_1\langle u_1, u_1\rangle + c_2\langle u_2, u_1\rangle = 0; \tag{10}$$

thus, since $\langle u_1, u_2\rangle = 0$ and $u_1$ is a unit vector, it follows from (10) that $c_1 = 0$. Similarly, $c_2 = 0$. We therefore get that the equation in (9) has only the trivial solution. Therefore, the set $\{u_1, u_2\}$ is linearly independent. Hence, in view of (6), $\{u_1, u_2\}$ is a basis for $\mathcal{I}_f$. □