Review Problems for Final Exam

1. Let W be a subspace of $\mathbb{R}^n$. Prove that $\text{span}(W) = W$.

2. Let S be linearly independent subset of $\mathbb{R}^n$. Suppose that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

3. Let W be a subspace of $\mathbb{R}^n$ with dimension $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W. Prove that there exist vectors $v_1, v_2, \ldots, v_{n-k}$ in $\mathbb{R}^n$ such that the set $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_{n-k}\}$ is a basis for $\mathbb{R}^n$.

4. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Prove that if $Ax = b$ has a solution x in $\mathbb{R}^n$, then $\langle b, v\rangle = 0$ for every v is the null space of A^T.

5. Let $R = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

   Compute the products RC and CR.

6. Let $A \in \mathbb{M}(m, n)$ and write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ denote the rows of A. Define $\mathcal{R}_A^{\perp}$ to be the set

   $$\mathcal{R}_A^{\perp} = \{w \in \mathbb{R}^n \mid R_i w = 0 \text{ for all } i = 1, 2, \ldots, m\};$$

   that is, $\mathcal{R}_A^{\perp}$ is the set of vectors in $\mathbb{R}^n$ which are orthogonal to the vectors $R_1^T, R_2^T, \ldots, R_m^T$ in $\mathbb{R}^n$.

   (a) Prove that $\mathcal{R}_A^{\perp}$ is a subspace of $\mathbb{R}^n$.

   (b) Prove that $\mathcal{R}_A^{\perp} = \mathcal{N}_A$.

   (c) Let v denote a vector in $\mathbb{R}^n$. Prove that if $v \in \mathcal{N}_A$ and $v^T \in \mathcal{R}_A$, then $v = \mathbf{0}$.

7. Let B be an $n \times n$ matrix satisfying $B^3 = 0$ and put $A = I + B$, where I denotes the $n \times n$ identity matrix. Prove that A is invertible and compute A^{-1} in terms of I, B and B^2.

8. Let $A, B \in \mathbb{M}(2, 2)$. Verify that $\det(AB) = \det(BA)$.

9. Let $A, B \in \mathbb{M}(2,2)$. Verify that $\det(A^T) = \det(A)$.

10. Given an $n \times n$ matrix $A = [a_{ij}]$, the trace of A, denoted $\operatorname{tr}(A)$, is the sum of the entries along the main diagonal of A; that is $\operatorname{tr}(A) = \displaystyle\sum_{i=1}^{n} a_{ii}$.

    Let A and B denote $n \times n$ matrices. Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

11. Let A and B be $n \times n$ matrices such that $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q.

    Prove that A and B have the same determinant and the same trace.

12. Let $A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$.

    (a) Find a basis $\mathcal{B} = \{v_1, v_2\}$ for $\mathbb{R}^2$ made up of eigenvectors of A.

    (b) Let Q be the 2×2 matrix $Q = [\ v_1 \quad v_2\ ]$, where $\{v_1, v_2\}$ is the basis of eigenvectors found in (a) above. Verify that Q is invertible and compute $Q^{-1}AQ$.

    (c) Use the result in part (b) above to find a formula for for computing A^k for every positive integer k. Can you say anything about $\displaystyle\lim_{k\to\infty} A^k$?

13. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in $\mathbb{R}^n$.

    (a) Suppose that the set of vectors $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is a linearly independent set of vectors in $\mathbb{R}^m$. Prove that the set S must be a linearly independent set in $\mathbb{R}^n$.

    (b) Is the converse of the statement in part (a) true? If not, produce a counter–example to show that the converse is generally false

14. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ denote a linear transformation. Let W denote the null space, $\mathcal{N}_T$, of T. Assume that W has dimension $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W and $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_{n-k}\}$ be a basis for $\mathbb{R}^n$. Prove that that the set $\{T(v_1), T(v_2), \ldots, T(v_{n-k})\}$ is a basis for $\mathcal{I}_T$, the image of T. Deduce the Dimension Theorem

    $$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

15. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation. Prove that if λ is an eigenvalue of T, then λ^k is an eigenvalue of T^k for every positive integer k. If μ is an eigenvalue of T^k, is $\mu^{1/k}$ always and eigenvalue of T?

16. Let $\mathcal{E} = \{e_1, e_2\}$ denote the standard basis in $\mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying: $f(e_1) = e_1 + e_2$ and $f(e_2) = 2e_1 - e_2$.

    Give the matrix representations for f and $f \circ f$ relative to $\mathcal{E}$.

17. A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows: Each vector $v \in \mathbb{R}^2$ is reflected across the y–axis, and then doubled in length to yield $f(v)$.

    Verify that f is linear and determine the matrix representation, M_f, for f relative to the standard basis in $\mathbb{R}^2$.

18. Find a 2×2 matrix A such that the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = Av$ maps the coordinates of any vector, relative to the standard basis in $\mathbb{R}^2$, to its coordinates relative the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

19. Let u_1 and u_2 denote a unit vector in $\mathbb{R}^3$ that are orthogonal to each other; i.e., $\langle u_1, u_2 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$.

    Define $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $f(v) = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$ for all $v \in \mathbb{R}^3$.

    (a) Use the Dimension Theorem to compute $\dim(\mathcal{N}_f)$.

    (b) Show that $v - f(v)$ is orthogonal to every vector w in the image of f.

    (c) Show that $f(v)$ gives the point in the plane spanned by u_1 and u_2 that is the closest to v in $\mathbb{R}^3$.