Assignment #1

Due on Thursday, September 8, 2016

Read Chapter 2, *An Example from Statistical Inference,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 3.1, *Sample Spaces and σ–fields,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 3.2, *Some Set Algebra,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 3.3, *More on σ–fields,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 1.4 on *Set Theory* in DeGroot and Schervish.

Background and Definitions

- A σ-field, $\mathcal{B}$, is a collection of subsets of a sample space $\mathcal{C}$, referred to as **events**, which satisfy:

  (1) $\emptyset \in \mathcal{B}$ ($\emptyset$ denotes the empty set)

  (2) If $E \in \mathcal{B}$, then its complement, E^c, is also an element of $\mathcal{B}$.

  (3) If $(E_1, E_2, E_3 \ldots)$ is a sequence of events, then

  $$E_1 \cup E_2 \cup E_3 \cup \ldots = \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.$$

- Let $\mathcal{S}$ denote a collection of subsets of a sample space $\mathcal{C}$. The σ–field generated by $\mathcal{S}$, denoted by $\mathcal{B}(\mathcal{S})$, is the smallest σ–field in $\mathcal{C}$ which contains $\mathcal{S}$.

- $\mathcal{B}_o$ denotes the Borel σ–field of the real line, $\mathbb{R}$. This is the σ–field generated by the semi–infinite intervals

  $$(-\infty, b], \qquad \text{for } b \in \mathbb{R}.$$

Do the following problems

1. Let $\mathcal{C}$ denote a sample space and A be a subset of $\mathcal{C}$. Establish the following set theoretic identities, where $\emptyset$ denotes the empty set. Justify your steps.

   (a) $A \cap \emptyset = \emptyset$ (b) $A \cup \emptyset = A$

2. Let $\mathcal{C}$ denote a sample space and A and B denote subsets of $\mathcal{C}$. Establish the following set theoretic identities:

   (a) $(A^c)^c = A$,

   (b) $(A \cup B)^c = A^c \cap B^c$;

   where A^c denote the complement of A.

3. Let $\mathcal{C}$ denote a sample space and A, B and C denote subsets of $\mathcal{C}$. Prove the following distributive properties:

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. Let A and B be subsets of the sample space $\mathcal{C}$. The *set difference* $A \backslash B$ is defined to be
$$A \backslash B = \{x \in A \mid x \notin B\};$$
   thus, $A \backslash B$ is a subset of A that contains those elements in A which are not in B.

   Prove that

   (a) $A \backslash B = A \cap B^c$,

   (b) $B \backslash (A \cap B) = A^c \cap B$

5. Suppose that $A \subseteq B$. Prove that $B^c \subseteq A^c$.

6. Let A, B and C be subsets of a sample space $\mathcal{C}$. Prove the following

   (a) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

   (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

7. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a σ–field of subsets of $\mathcal{C}$. Prove that if $\{E_1, E_2, E_3 \ldots\}$ is a sequence of events in $\mathcal{B}$, then
$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$
   Hint: Use De Morgan's Laws.

8. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a σ–field of subsets of $\mathcal{C}$. For fixed $B \in \mathcal{B}$ define the collection of subsets

$$\mathcal{B}_B = \{D \subseteq \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

Show that $\mathcal{B}_B$ is a σ–field.

Note: In this case, the complement of $D \in \mathcal{B}_B$ has to be understood as $B \backslash D$; that is, the complement relative to B. The σ–field $\mathcal{B}_B$ is the σ–field $\mathcal{B}$ restricted to B, or *conditioned on* B.

9. Let $\mathcal{S}$ denote the collection of all bounded, open intervals (a, b), where a and b are real numbers with $a < b$. Show that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o;$$

that is, the σ–field generated by bounded open intervals is the Borel σ–field.

Hints:

- We have already seen in the lecture that $\mathcal{B}_o$ contains all bounded open intervals.
- Observe also that the semi–infinite open interval (b, ∞) can be expressed as the union of the sequence of bounded intervals (b, k), for $k = 1, 2, 3, \ldots$

10. Show that for every real number a, the singleton $\{a\}$ is in the Borel σ–field $\mathcal{B}_o$.

Hint: Express $\{a\}$ as an intersection of a sequence of open intervals.