Assignment #3

Due on Thursday, September 22, 2016

Read Sections 3.5 on *Independent Events* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Sections 3.6 on *Conditional Probability* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 2.1 on *The Definition of Conditional Probability* in DeGroot and Schervish.

Read Section 2.2 on *Independent Events* in DeGroot and Schervish.

Do the following problems.

1. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ be a probability space. Prove that if E_1 and E_2 are independent events in $\mathcal{B}$, then so are E_1 and E_2^c.

   Hint: Observe that $E_1 \setminus E_2$ is a subset of E_1.

2. Let A and B denote events in a probability space $(\mathcal{C}, \mathcal{B}, \Pr)$.

   (a) If $A \subseteq B$ with $\Pr(B) > 0$, what is the value of $\Pr(A \mid B)$?

   (b) If A and B are disjoint events and $\Pr(B) > 0$, what is the value of the conditional probability $\Pr(A \mid B)$?

3. A box contains r red balls and b blue balls. One ball is selected at random and the color is observed. The ball is then returned to the the box and k additional balls of the same color are also put in the box. A second ball is the selected at random, its color is observed, and it is returned to the box with k additional balls of the same color. Each time another ball is selected, the process is repeated. If four balls are selected, what is the probability that the first three balls will be red and the fourth one will be blue?

4. For any three events A, B and D, such that $\Pr(D) > 0$, prove that

$$\Pr(A \cup B \mid D) = \Pr(A \mid D) + \Pr(B \mid D) - \Pr(A \cap B \mid D).$$

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space. Three events, E_1, E_2 and E_3, are said to be *mutually independent* is they are pairwise independent, and

$$\Pr(E_1 \cap E_2 \cap E_3) = \Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(E_3).$$

Let $\mathcal{C} = \{1, 2, 3, 4\}$ and $\mathcal{B}$ be the set of all subsets of $\mathcal{C}$. Define a probability on $\mathcal{B}$ using the equal likelihood assumption; that is,

$$\Pr(c) = \frac{1}{4}, \text{ for all } c \in \mathcal{C}.$$

Put $E_1 = \{1, 2\}$, $E_2 = \{1, 3\}$ and $E_3 = \{2, 3\}$.

Verify that E_1, E_2 and E_3 are pairwise independent, but are not mutually independent.

6. Toss a balanced die twice in a row. Let E_1 denote the event that the first toss yields either a 1, or a 2, or a 3; E_2 the event that the first toss yields a 3, or a 4, or a 5; and E_3 the event that the sum of the outcomes of the two tosses is 9.

   (a) Verify that

$$\Pr(E_1 \cap E_2 \cap E_3) = \Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(E_3).$$

   (b) Show that E_1, E_2 and E_3 are not pairwise independent.

7. An experiment consists of dawning 2 balls at random from a bin containing 3 red balls and 2 green balls (the balls are not replaced between draws). Let A be the event that both balls are red, B the event that both balls are green, and C the event that the first ball is red.

   (a) Compute the probabilities of A, B, and C.

   (b) Compute $\Pr(A \mid B)$ and $P(B \mid A)$.

   (c) Are the events A and B independent? Justify your answer.

8. A researcher is studying the prevalence of three health risk factors, denoted A, B and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only that risk factor and not the others. For any two of the three factors, the probability is 0.12 a

woman has exactly those two risk factors, but not the other. Assume that the probability that a woman has all three risk factors, given that she has A and B, is 1/3. Compute the probability that a woman has none of the three risk factors, given that she does not have risk factor A.

9. A box contains three coins: two regular coins and one fake, two–headed coin.

   (a) You pick a coin at random and toss it. What is the probability that it lands heads up?

   (b) You pick a coin at random and toss it, and get heads. What is the probability that it is the two–headed coin?

10. **The Monty Hall Problem**. In a game show, suppose there are three curtains. Behind one curtain is a nice prize while behind the other two there are worthless prizes. A contestant selects one curtain at random, and then Monte Hall (the game show host) opens one the other two curtains to reveal a worthless prize. Hall then expresses the willingness to trade the curtain that the contestant has selected for the other curtain that has not been opened. Should the contestant switch curtains or stick with the one that she has? If she sticks with the one she has then the probability of winning the prize is 1/3. Hence, to answer this question, you must determine the probability that she wins the prize given that she switches.