Assignment #4

Due on Thursday, October 6, 2016

Read Section 4.1 *Definition of Random Variable* in the class lecture notes at
http://pages.pomona.edu/~ajr04747/

Read Section 4.2 *Distribution Functions* in the class lecture notes at
http://pages.pomona.edu/~ajr04747/

Read Section 3.1 on *Random Variables and Discrete Distributions* in DeGroot and Schervish.

Read Section 3.2 on *Continuous Distributions* in DeGroot and Schervish.

Do the following problems.

1. For each of the following, find the value of the constant c for which the given function, $p(x)$, is the probability mass function (pmf) of some discrete random variable.

   (a) $p(x) = c\left(\dfrac{2}{3}\right)^x$, for $x = 1, 2, 3, \ldots$ and zero elsewhere.

   (b) $p(x) = cx$ for $x = 1, 2, 3, 4, 5$, and zero otherwise.

2. Suppose that two balanced dice are rolled, and let X denote the absolute value of the difference between the two numbers that appear. Determine and sketch the pmf of X.

3. Suppose that a box contains seven red balls and three blue balls. If five of them are selected at random, without replacement, determine the pmf of the number of red balls that will be obtained.

4. A civil engineer is studying a left–turn lane that is long enough to hold 7 cars. Let X denote the number of cars left in the lane at the end of randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x + 1)(8 - x)$ for $x = 0, 1, \ldots, 7$ (the possible values of X).

   (a) Find the pmf for X.

   (b) Find the probability that X will be at least 5.

5. Select five cards at random and without replacement from an ordinary deck of playing cards. Let X denote the number of hearts in the five cards.

   (a) Find the probability mass function (pmf) of X. Denote it by $p(x)$.

   (b) Determine $\Pr(X \leqslant 1)$.

   (c) Find the cumulative distribution function, $F(x) = \Pr(X \leqslant x)$, and sketch its graph along with that of $p(x)$.

6. Suppose the pdf of a random variable X is as follows:

$$f(x) = \begin{cases} \dfrac{4}{3}(1 - x^3) & \text{for } \ 0 < x < 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

   Sketch the pdf and determine the values of the following probabilities:

   (a) $\Pr\left(X < \dfrac{1}{2}\right)$

   (b) $\Pr\left(\dfrac{1}{4} < X < \dfrac{3}{4}\right)$

   (c) $\Pr\left(X > \dfrac{1}{3}\right)$

7. Suppose the pdf of a random variable is as follows:

$$f(x) = \begin{cases} cx^2 & \text{for } \ 1 \leqslant x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of c and sketch the pdf.

   (b) Find the value of $\Pr(X > 3/2)$.

8. Let $\mathcal{C} = \{x \in \mathbb{R} \mid 0 < x < \infty\}$ and $\mathcal{B}$ denote the Borel sets in $\mathcal{C}$. Let the pdf of a random variable, X, defined on $\mathcal{C}$ be given by

$$f_X(x) = e^{-x} \quad \text{for all } \ x > 0.$$

   Let $E_k = \{x \in \mathcal{C} \mid 2 - 1/k < x \leqslant 3\}$ for $k = 1, 2, 3, \ldots$

   Compute $\Pr(E_n)$ for all n, and $\lim\limits_{n\to\infty} \Pr(E_n)$.

9. A point is selected at random form the sample space $\mathcal{C} = \{x \in \mathbb{R} \mid 0 < x < 10\}$. For any Borel subset $E \subseteq \mathcal{C}$ the probability of E is defined to be
$$\Pr(E) = \int_E \frac{1}{10}\,\mathrm{d}x.$$
Define $X \colon \mathcal{C} \to \mathbb{R}$ to be
$$X(x) = x^2 \quad \text{for all} \quad x \in \mathcal{C}.$$
Find the cumulative distribution function and the probability density function of X.

10. A *median* of the distribution of a random variable X is a value m for x such that
$$\Pr(X < m) \leqslant \frac{1}{2} \quad \text{and} \quad \Pr(X \leqslant m) \geqslant \frac{1}{2}.$$
If there is only one such value m, it is called *the median of the distribution*.

Suppose the pdf of a random variable X is given by the function
$$f(x) = \begin{cases} \dfrac{1}{8}x & \text{for } 0 \leqslant x \leqslant 4, \\ \\ 0 & \text{otherwise.} \end{cases}$$
Compute a median for the distribution of X. Is it <u>the</u> median of the distribution?