Assignment #8

Due on Thursday, November 10, 2016

Read Section 6.1 on the *Definition of the Joint Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 6.2 on *Marginal Distributions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 6.3 on the *Independent Random Variables* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.4 on *Bivariate Distributions* in DeGroot and Schervish.

Read Section 3.5 on *Marginal Distributions* in DeGroot and Schervish.

Read Section 3.9 on *Functions of Two or More Random Variables* in DeGroot and Schervish.

Do the following problems.

1. Suppose that in an electric display sign there are three light bulbs in the first row and four light bulbs in the second row. Let X denote the number of bulbs in the first row that will be burned out at a specified time t, and let Y denote the number of bulbs in the second row that will be burned out at the same time t. Suppose that the joint pmf of X and Y is as specified in Table 1:

$X\backslash Y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

Determine each of the following probabilities:

(a) $\Pr(X = 2)$ (b) $\Pr(Y \geqslant 2)$ (c) $\Pr(X \leqslant 2 \text{ and } Y \leqslant 2)$

(d) $\Pr(X = Y)$ (e) $\Pr(X > Y)$

2. Suppose that X and Y have a continuous joint distribution for which the pdf is defined as follows: $f(x,y) = \begin{cases} cy^2 & \text{for } 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$

   Determine

   (a) the value of c; (b) $\Pr(X+Y>2)$; (c) $\Pr(Y<1/2)$;
   (d) $\Pr(X \leqslant 1)$; (e) $\Pr(X=3Y)$.

3. Suppose a point X is chosen at random from a region S in the xy–plane containing all points (x,y) such that $x \geqslant 0$, $y \geqslant 0$, and $4y + x \leqslant 4$.

   (a) Determine the joint pdf of X and Y.
   (b) Suppose that S_o is a subset of the region S having area α, and determine $\Pr[(X,Y) \in S_o]$.

4. Suppose that X and Y have a discrete distribution for which the joint pmf is defined as follows:

$$p_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{30}(x+y) & \text{for } x = 0,1,2 \text{ and } y = 0,1,2,3, \\ \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the marginal pmfs of X and Y.
   (b) Are X and Y independent?

5. Suppose the joint pdf of X and Y is as follows:

$$f_{(X.Y)}(x,y) = \begin{cases} \dfrac{15}{4}x^2 & \text{for } 0 \leqslant y \leqslant 1 - x^2 \\ \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the marginal pdfs of X and Y.
   (b) Are X and Y independent?

6. Suppose X and Y are independent and let $g_1(X)$ and $g_2(Y)$ be functions for which $E(g_1(X)g_2(Y))$ exists. Show that

$$E(g_1(X)g_2(Y)) = E(g_1(X)) \cdot E(g_2(Y))$$

Conclude therefore that if X and Y are independent and $E(|XY|)$ is finite, then

$$E(XY) = E(X) \cdot E(Y).$$

7. Suppose X and Y are independent random variables for which the moment generating functions exist on some common interval of values of t. Show that

$$\psi_{X+Y}(t) = \psi_X(t) \cdot \psi_Y(t)$$

for t is the given interval.

8. **Definition of Covariance**. Given random variables X and Y, put $\mu_X = E(X)$ and $\mu_Y = E(Y)$. The *covariance* of X and Y, denoted $\text{Cov}(X, Y)$ is defined by

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)], \tag{1}$$

provided that the expectation in (1) exists.

Let X and Y denote random variables for which $\text{var}(X)$ and $\text{var}(Y)$ exist; that is, $\text{var}(X) < \infty$ and $\text{var}(Y) < \infty$. Show that $\text{Cov}(X, Y)$ exists.

Suggestion: Use the inequality

$$|ab| \leqslant \frac{1}{2}(a^2 + b^2),$$

for all real numbers a and b.

9. Assume that X and Y have joint pdf

$$f_{(X,Y)}(x, y) = \begin{cases} 2xy + \dfrac{1}{2}, & \text{for } 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Compute the covariance of X and Y.

10. Let X and Y denote random variables with finite variance.

    (a) Derive the identity

    $$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y).$$

    (b) Show that if X and Y are independent, then $\text{Cov}(X, Y) = 0$.