Assignment #9

Due on Thursday, November 17, 2016

Read Section 6.3 on the *Independent Random Variables* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 7.1 on *The Normal Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.6 on *The Normal Distributions* in DeGroot and Schervish.

Do the following problems.

1. Suppose that $X \sim \text{Normal}(\mu, \sigma^2)$ and define $Z = \dfrac{X-\mu}{\sigma}$.

   Prove that $Z \sim \text{Normal}(0,1)$

2. (*The Chi–Square Distribution*) Let $X \sim \text{Normal}(0,1)$ and define $Y = X^2$. Compute the pdf, $f_{_Y}$, of Y.

   The distribution of Y is called the *Chi–Square distribution with one degree of freedom;* we write $Y \sim \chi^2(1)$.

3. (*Moment Generating Function of the Chi–Square Distribution*) Assume that $Y \sim \chi^2(1)$. Compute the mgf, $\psi_{_Y}$, of Y by computing $E(e^{tY}) = E(e^{tX^2})$, where $X \sim \text{Normal}(0,1)$.

   Use the mgf of Y to compute $E(Y)$ and $\text{var}(Y)$.

4. Let Y_1 and Y_2 denote two independent random variables such that $Y_1 \sim \chi^2(1)$ and $Y_2 \sim \chi^2(1)$. Define $X = Y_1 + Y_2$. Use the mgf of the $\chi^2(1)$ distribution found in Problem 3 to compute the mgf of X. Give the distribution of X.

5. Let X_1 and X_2 denote independent, $\text{Normal}(0,\sigma^2)$ random variables, where $\sigma > 0$. Define the random variables

   $$\overline{X} = \frac{X_1 + X_2}{2} \quad \text{and} \quad Y = \frac{(X_1 - X_2)^2}{2\sigma^2}.$$

   Determine the distributions of $\overline{X}$ and Y.

   Suggestion: To obtain the distribution for Y, first show that

   $$\frac{X_1 - X_2}{\sqrt{2}\,\sigma} \sim \text{Normal}(0,1).$$

6. Let X and Y be independent Normal$(0, 1)$ random variables.

   Compute $\Pr(X^2 + Y^2 < 1)$.

7. Let $X_1, X_2, X_3, \ldots, X_n$ be independent identically distributed Normal$(0, 1)$ random. Define
$$Y = X_1 + X_2 + \cdots + X_n.$$
   Use moment generating functions to determine the distribution of Y.

   Compute $E(Y)$ and $\text{var}(Y)$.

8. Let $X_1, X_2, X_3, \ldots, X_n$ be independent identically distributed Normal$(0, 1)$ random. Define
$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$
   Use moment generating functions to determine the distribution of $\overline{X}$.

   Compute $E(\overline{X})$ and $\text{var}(\overline{X})$.

9. Let X denote a nonnegative random variable. Assume that $\ln(X)$ has a standard normal distribution. Compute the pdf of X.

10. Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$.

    Let X_1 denote the measurement made by the first instrument and X_2 the measurement made by the second instrument. Assume that X_1 and X_2 are independent random variables, and let $X = \dfrac{X_1 + X_2}{2}$, the average of the two instruments.

    (a) Determine the distribution of X.

    (b) Compute the probability that their average of the two measurements is within $0.005h$ of the height of the tower?