Solutions to Exam 1

1. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space.

   (a) State precisely what it means for $E_1, E_2, E_3 \in \mathcal{B}$ to be mutually exclusive.

   Answer: E_1, E_2 and E_3 are mutually exclusive means that they are pairwise disjoint; that is,

   $$E_i \cap E_j = \emptyset, \qquad \text{for } i \neq j.$$

   □

   (b) State precisely what it means for $E_1, E_2, E_3 \in \mathcal{B}$ to be mutually independent.

   Answer: E_1, E_2 and E_3 are mutually independent means that they are pairwise independent; that is,

   $$\Pr(E_i \cap E_j) = \Pr(E_i) \cdot \Pr(E_j), \qquad \text{for } i \neq j,$$

   and

   $$\Pr(E_1 \cap E_2 \cap E_3) = \Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(E_3).$$

   □

2. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B be elements in the σ–Field $\mathcal{B}$. Recall that $A \setminus B = \{x \in A \mid x \notin B\}$. Use the additivity property of probability to derive the fact: $\Pr(A \setminus B) = \Pr(A) - \Pr(A \cap B)$.

   Provide reasons for the steps in your derivation.

   Solution: Observe that

   $$A = (A \cap B) \cup (A \backslash B),$$

   where $A \cap B$ and $A \backslash B$ are disjoint. Hence, by the additivity property of probability,

   $$\Pr(A) = \Pr(A \cap B) + \Pr(A \backslash B),$$

   from which we get that

   $$\Pr(A \backslash B) = \Pr(A) - \Pr(A \cap B),$$

   which was to be shown. □

3. Consider a fair six–sided die where one side is labeled 1, and the others are all labeled 0. The die is rolled twice in a row. Let A denote the event that the first die is a 1, B the event that the second die is a 0.

   (a) Give the elements in the sample space for this experiment.

   Solution: The sample space, $\mathcal{C}$ for this experiment is

   $$\mathcal{C} = \{(0,0), (0,1), (1,0), (1,1)\},$$

   where the pair (m, n) indicates that event that the number m comes up in the first toss of the die and the number n comes up in the second toss. □

   (b) Give the probability, Pr, associated with each of the sample points.

   Solution: Compute

   $$\begin{aligned} \Pr((0,0)) &= \frac{5}{6}\cdot\frac{5}{6} = \frac{25}{36}, \\ \Pr((0,1)) &= \frac{5}{6}\cdot\frac{1}{6} = \frac{5}{36}, \\ \Pr((1,0)) &= \frac{1}{6}\cdot\frac{5}{6} = \frac{5}{36}, \\ \Pr((1,1)) &= \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}, \end{aligned}$$

   where we have used the fact that the outcomes of the tosses of the die are independent events. □

   (c) Compute $\Pr(A)$ and $\Pr(B)$.

   Solution: Observe that $A = \{(1,0), (1,1)\}$ and $B = \{(0,0), (1,0)\}$. Then, using the probabilities computed in the previous part,

   $$\Pr(A) = \Pr((1,0)) + \Pr((1,1)) = \frac{5}{36} + \frac{1}{36} = \frac{1}{6},$$

   and

   $$\Pr(B) = \Pr((0,0)) + \Pr((1,0)) = \frac{25}{36} + \frac{5}{36} = \frac{5}{6}.$$

   □

4. For the probability space $(\mathcal{C}, \mathcal{B}, \Pr)$ and events A and B defined in Problem 3,

(a) compute $\Pr(A \cap B)$ and $\Pr(A \mid B)$.

Solution: First, compute $A \cap B = \{(1,0)\}$ and use the probabilities computed in part (b) of the previous problem to get

$$\Pr(A \cap B) = \Pr((1,0)) = \frac{5}{36}.$$

Next, compute

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\frac{5}{36}}{\frac{5}{6}} = \frac{1}{6}.$$

□

(b) Are the events A and B independent? Explain your answer.

Solution: Observe that $\Pr(A \mid B) = \Pr(A)$. Hence, A and B are independent. □

5. A box contains two coins: one of the coins is a fair coin and the other one is a two–headed coin.

(a) You pick a coin at random and toss it. What is the probability that it lands heads up?

Solution: Let R denote the event that we pick the fair coin, and F the event that we pick the false coin. Then, by the Law of Total Probability, the probability of a head, H, in the flip of the coin that we picked up is

$$\Pr(H) = \Pr(R) \cdot \Pr(H \mid R) + \Pr(F) \cdot \Pr(H \mid F),$$

where

$$\Pr(R) = \frac{1}{2}, \quad \Pr(F) = \frac{1}{2}, \quad \Pr(H \mid R) = \frac{1}{2}, \text{ and } \Pr(H \mid F) = 1.$$

Then,

$$\Pr(H) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}.$$

□

(b) You pick a coin at random and toss it, and get heads. What is the probability that it is the two–headed coin?

***Solution**:* We want $\Pr(F \mid H)$, which we can compute as

$$\Pr(F \mid H) \;=\; \frac{\Pr(F \cap H)}{\Pr(H)},$$

where

$$\Pr(H) = \frac{3}{4},$$

as computed in part (a) of this problem, and

$$\Pr(F \cap H) = \Pr(F) \cdot \Pr(H \mid F) = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Hence,

$$\Pr(F \mid H) = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}.$$

□