Review Problems for Exam 2

1. **The Poisson Random Process Revisited**. We saw in class and in the lecture notes online how to define a Poisson random process, $\{M(t) \mid t \geqslant 0\}$, to model occurrences of events at random points in time (e.g., occurrences of mutations in a bacterial colony). Here $M(t)$ counts the number of occurrences in the time interval $[0, t]$. This continuous–time random process may also be defined as one satisfying the following axioms:

   (i) $M(0) = 0$.

   (ii) The number of events that occur in disjoint time intervals are independent; in symbols, for $t_1 < t_2 < t_3 < t_4$,

   $$M(t_2) - M(t_1) \text{ and } M(t_4) - M(t_3) \text{ are independent random variables.}$$

   (iii) The number of occurrences within a time interval depends only on the length of the time interval; in symbols, for all $t, s > 0$, $M(t+s) - M(t)$ depends only on s, so that

   $$\Pr[M(t+s) - M(t) = k] = \Pr[M(s) - M(0) = k], \quad \text{for all } k.$$

   (iv) $\Pr[M(\Delta t) = 1] = \lambda \Delta t + o(\Delta t)$.

   (v) $\Pr[M(\Delta t) \geqslant 2] = o(\Delta t)$.

   The notation $o(h)$ in (iv) and (v) is defined as follows: We say that an expression, $f(h)$, is $o(h)$ as $h \to 0$ iff $\displaystyle\lim_{h \to 0} \frac{f(h)}{h} = 0$.

   The constant λ in (iv) is called the rate of the process.

   Set
   $$P_m(t) = \Pr[M(t) = m], \quad \text{for } m = 0, 1, 2, 3\ldots, \text{ and } t \geqslant 0.$$

   Use the axioms (i)–(v) to prove the following assertions.

   (a) For $t, s > 0$,
   $$P_0(t+s) = P_0(t) \cdot P_0(s). \tag{1}$$

   Suggestion: Consider the event $[M(t) = 0, M(t+s) - M(t) = 0]$ or

   $$[M(t) = 0] \cap [M(t+s) - M(t) = 0].$$

(b) Use (1) and axioms (iv) and (v) to derive the differential equation

$$\frac{dP_0}{dt} = -\lambda P_0(t). \tag{2}$$

Suggestion: Verify that

$$P_0(t + \Delta t) - P_0(t) = -\lambda P_0(t)\Delta t + o(\Delta t).$$

(c) Solve the differential equation in (2) subject to the initial condition in (i) to obtain and expression for $P_0(t)$ for all $t \geqslant 0$.

(d) Let T_1 denote the time of the first occurrence, and, for $n \geqslant 2$, let T_n denote the time elapsed between the $(n-1)^{\text{st}}$ occurrence and the the n^{th} occurrence. The sequence (T_n) is called the sequence of interarrival times. Give the distribution for each of the random variables T_n.

Suggestion: We have already done the derivation of the distribution for T_1 in the class notes and assignments. Please, present the derivation here as well.

For $n = 2$, consider the conditional probabilities

$$\Pr[T_2 > s + t \mid T_1 = s] \text{ and } \Pr[M(s+t) - M(s) = 0 \mid M(s) = 1].$$

(e) Let S_n denote the time of occurrence of the n^{th} event, so that

$$S_n = \sum_{k=1}^{n} T_k, \qquad \text{for } n = 1, 2, 3, \ldots$$

Show that, for each $n = 1, 2, 3, \ldots$, S_n is a continuous random variable with density function given by

$$f_{S_n}(s) = \lambda e^{-\lambda s}\frac{(\lambda s)^{n-1}}{(n-1)!}, \qquad \text{for } n = 1, 2, 3, \ldots \tag{3}$$

Suggestion: Proceed by induction on n. The base case, $n = 1$, has already been established. For the case $n = 2$, so that $S_2 = T_1 + T_2$, use the fact that, since T_1 and T_2 are independent random variables, the distribution of S_2 is given by the convolution formula

$$f_{S_n}(s) = f_{T_1} * f_{T_2}(s) = \int_{-\infty}^{\infty} f_{T_1}(\tau) f_{T_2}(s - \tau)\, d\tau \tag{4}$$

Note that the convolution formula in (4) applies to any sum of independent, continuous random variables.

(f) Use the result in (3) to derive the formula

$$P_m(t) = \frac{(\lambda t)^m}{m!}\, e^{-\lambda t}, \qquad \text{for } m = 0, 1, 2, 3, \ldots, \text{ and } t \geqslant 0.$$

Suggestion: Consider the events

$$[M(t) \geqslant n] \quad \text{and} \quad [S_n \leqslant t]$$

and note that

$$[M(t) = n] = [n \leqslant M(t) < n+1]$$

(g) Suppose that exactly one event has occurred in the time interval $[0, \tau]$. We consider the time of occurrence, T_1, of that event. Compute the conditional probability

$$\Pr[T_1 \leqslant t \mid M(\tau) = 1], \text{ for } 0 < t < \tau.$$

Suggestion: Consider the events

$$[T_1 \leqslant t, M(\tau) = 1] \quad \text{and} \quad [M(t) = 1, M(\tau) - M(t) = 0],$$

for $0 < t < \tau$.

2. **The Error function**, Erf: $\mathbb{R} \to \mathbb{R}$, is defined by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds, \qquad \text{for } x \in \mathbb{R}. \tag{5}$$

Use the fact that

$$\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$$

to deduce that

(a) $\lim\limits_{x\to\infty} \mathrm{Erf}(x) = 1$; and

(b) $\lim\limits_{x\to-\infty} \mathrm{Erf}(x) = -1$.

3. **Solving the Heat Equation**. In this problem we compute a solution of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0; \\ u(x, 0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

where

$$f(x) = \begin{cases} 1, & \text{if } x \leqslant 0; \\ 0, & \text{if } x > 0. \end{cases} \tag{7}$$

(a) Use the heat kernel to give a solution of the IVP (6).

(b) Use a mathematical software package to sketch the graph of $x \mapsto u(x,t)$ for several values of $t > 0$, where $u(x,t)$ is the solution of the initial value problem (6) with initial condition in (7) obtained in part (a).

(c) Let $u(x,t)$ be the solution to the initial value problem (6) with initial condition in (7) obtained in part (a). Compute the following

  (i) $\lim\limits_{t\to 0^+} u(x,t)$, for $x = 0$ and for $x \neq 0$.

  (ii) $\lim\limits_{x\to 0} u(x,t)$, for all $t > 0$.

(d) Let $u(x,t)$ be the solution of the initial value problem (6) with initial condition in (7) obtained in part (a). Compute the following

  (i) $\lim\limits_{t\to \infty} u(x,t)$, for $x = 0$ and for $x \neq 0$.

  (ii) $\lim\limits_{x\to \infty} u(x,t)$, for all $t > 0$.

  (iii) $\lim\limits_{x\to -\infty} u(x,t)$, for all $t > 0$.