Solutions to Assignment #13

1. Use the method of integrating factor discussed in Section 4.8.5 in the class lecture notes to find the general solution of the linear, first order differential equation with constant coefficients

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constant with $a \neq 0$.

Compare your result to what you obtain when you solve (1) via separation of variables.

***Solution**:* Write the equation (1) in the form

$$\frac{dy}{dt} - ay = b, \tag{2}$$

and multiply by e^{-at} on both sides (2) to obtain

$$e^{-at}\frac{dy}{dt} - ae^{-at}y = be^{-at},$$

which can be written as

$$\frac{d}{dt}\left[e^{-at}y\right] = be^{-at}, \tag{3}$$

by virtue of the product rule. Integrating on both sides of (3) with respect to t we obtain

$$e^{-at}y = -\frac{b}{a}e^{-at} + c, \tag{4}$$

for arbitrary c. Next, multiply both sides of the equation in (4) by e^{at} to obtain the general solution

$$y(t) = -\frac{b}{a} + c\, e^{at},$$

which is the same solution given by the method of separation of variables. $\square$

2. Use the method of integrating factor discussed in Section 4.8.5 to find the general solution of the linear, first order differential equation

$$\frac{dy}{dt} = 2ty + t. \tag{5}$$

***Solution**:* Rewrite the equation in (5) as

$$\frac{dy}{dt} - 2ty = t, \tag{6}$$

and multiply on both sides of (6) by the integrating factor $\mu(t) = e^{-t^2}$ to obtain

$$e^{-t^2}\frac{dy}{dt} - 2te^{-t^2}y = te^{-t^2}. \tag{7}$$

Next, use the product rule to see that (7) can be written as

$$\frac{d}{dt}\left[e^{-t^2}y\right] = te^{-t^2}. \tag{8}$$

Integrate on both sides of (8) with respect to t to obtain

$$e^{-t^2}y = -\frac{1}{2}e^{-t^2} + c, \tag{9}$$

for arbitrary c. Finally, multiply on both sides of (9) by e^{t^2} to solve for y,

$$y(t) = -\frac{1}{2} + c\, e^{t^2}, \quad \text{for all } t.$$

$\square$

3. Find the general solution of the linear, first order differential equation

$$\frac{dy}{dt} = y + e^{2t}. \tag{10}$$

***Solution**:* Rewrite the equation in (10) as

$$\frac{dy}{dt} - y = e^{2t}, \tag{11}$$

and multiply on both sides of (11) by the integrating factor $\mu(t) = e^{-t}$ to obtain

$$e^{-t}\frac{dy}{dt} - e^{-t}y = e^{t},$$

which can be written as

$$\frac{d}{dt}\left[e^{-t}y\right] = e^{t}, \tag{12}$$

by virtue of the product rule. Integrating on both sides of (12) with respect to t yields

$$e^{-t}y = e^{t} + c. \tag{13}$$

for arbitrary c. Solving for y in (13) yields the general solution

$$y(t) = e^{2t} + c\, e^{t}, \quad \text{for all } t.$$

$\square$

4. Find the general solution of the linear, first order differential equation

$$\frac{dy}{dt} = -\frac{1}{2t}y + t, \quad \text{for } t > 0. \tag{14}$$

Solution: Rewrite the equation in (14) as

$$\frac{dy}{dt} + \frac{1}{2t}y = t, \quad \text{for } t > 0. \tag{15}$$

In this case, an integrating factor is provided by

$$\begin{aligned} \mu(t) &= \exp\left(\int \frac{1}{2t}\, dt\right) \\ &= \exp\left(\frac{1}{2}\ln t\right) \\ &= \exp\left(\ln\sqrt{t}\right) \\ &= \sqrt{t}, \end{aligned}$$

for $t > 0$. Thus, multiplying on both sides of (15) by $\mu(t) = \sqrt{t}$, for $t > 0$, yields

$$\sqrt{t}\frac{dy}{dt} + \frac{1}{2\sqrt{t}}y = t^{3/2}, \quad \text{for } t > 0.$$

which can be written as

$$\frac{d}{dt}\left[\sqrt{t}y\right] = t^{3/2}, \quad \text{for } t > 0, \tag{16}$$

by virtue of the product rule.

Next, integrate (16) with respect to t to get

$$\sqrt{t}y = \frac{2}{5}t^{5/2} + c, \quad \text{for } t > 0, \tag{17}$$

and for arbitrary c.

Finally, solving for y in (17) yields the general solution

$$y(t) = \frac{2}{5}t^2 + \frac{c}{\sqrt{t}}, \quad \text{for } t > 0. \tag{18}$$

□

5. Solve the initial value problem

$$\frac{dy}{dt} = -\frac{1}{2t}y + t, \qquad \text{for } t > 0, \qquad y(1) = 0. \tag{19}$$

Solution: The general solution of the differential equation in (19) is given in (18). Using the initial condition the yields

$$\frac{2}{5} + c = 0,$$

which yields $c = -\dfrac{2}{5}$. Substituting this value of c into the formula for $y(t)$ in (18) then yields

$$y(t) = \frac{2}{5}\left(t^2 - \frac{1}{\sqrt{t}}\right), \qquad \text{for } t > 0.$$

□