Solutions to Assignment #15

1. Evaluate the integral $\displaystyle\int \frac{y^2+1}{y^3-4y^2+y+6}\,dy$, by first finding constants A, B and C such that

$$\frac{y^2+1}{y^3-4y^2+y+6} = \frac{A}{y-2} + \frac{B}{y+1} + \frac{C}{y-3}. \qquad (1)$$

***Solution**:* It follows from (1) that

$$\int \frac{y^2+1}{y^3-4y^2+y+6}\,dy = A\ln|y-2| + B\ln|y+1| + C\ln|y-3| + c, \qquad (2)$$

for arbitrary constant c. In order to determine the values of A, B and C, we multiply on both sides of (1) by $(y-2)(y+1)(y-3)$ to obtain

$$y^2+1 = A(y+1)(y-3) + B(y-2)(y-3) + C(y-2)(y+1),$$

or

$$y^2+1 = A(y^2-2y-3) + B(y^2-5y+6) + C(y^2-y-2),$$

or

$$y^2+1 = (A+B+C)y^2 + (-2A-5B-C)y + (-3A+6B-2C), \qquad (3)$$

after simplifying. Equating corresponding coefficients of the polynomials in (3) yields the system

$$\begin{cases} A+B+C &= 1 \\ -2A-5B-C &= 0 \\ -3A+6B-2C &= 1. \end{cases} \qquad (4)$$

Solving the system in (4) yields

$$A = -\frac{5}{3}, \quad B = \frac{1}{6} \quad \text{and} \quad C = \frac{5}{2}. \qquad (5)$$

It then follows from (1)and (5) that

$$\int \frac{y^2+1}{y^3-4y^2+y+6}\,dy = -\frac{5}{3}\ln|y-2| + \frac{1}{6}\ln|y+1| + \frac{5}{2}\ln|y-3| + c.$$

□

2. Evaluate the integral $\displaystyle\int \frac{y^2 - y + 6}{y^3 - 5y^2 + y - 5}\, dy$, by first finding constants A, B and C such that

$$\frac{y^2 - y + 6}{y^3 - 5y^2 + y - 5} = \frac{A}{y-5} + \frac{By + C}{y^2 + 1}. \tag{6}$$

***Solution**:* In order to determine the values of A, B and C, we multiply on both sides of (6) by $(y-5)(y^2+1)$ to obtain

$$y^2 - y + 6 = A(y^2 + 1) + (By + C)(y - 5),$$

or

$$y^2 - y + 6 = Ay^2 + A + By^2 - 5By + Cy - 5C,$$

or

$$y^2 - y + 6 = (A + B)y^2 + (-5B + C)y + (A - 5C), \tag{7}$$

after simplifying. Equating corresponding coefficients of the polynomials in (7) yields the system

$$\begin{cases} A + B &= 1 \\ -5B + C &= -1 \\ A - 5C &= 6. \end{cases} \tag{8}$$

Solving the system in (8) yields

$$A = 1, \quad B = 0 \quad \text{and} \quad C = -1. \tag{9}$$

We then obtain from (6)

$$\frac{y^2 - y + 6}{y^3 - 5y^2 + y - 5} = \frac{1}{y-5} - \frac{1}{y^2 + 1}. \tag{10}$$

Integration on both sides of (10) then yields

$$\int \frac{y^2 - y + 6}{y^3 - 5y^2 + y - 5}\, dy = \ln|y - 5| - \arctan(y) + c,$$

for arbitrary constant c. □

3. Solve the initial value problem

$$\frac{dy}{dt} = y - \frac{1}{3}y^2, \qquad y(0) = 1, \tag{11}$$

and sketch the solution.

***Solution**:* First, rewrite the differential equation in (11) as

$$\frac{dy}{dt} = -\frac{1}{3}y(y-3),$$

and the separate variables to get

$$\int \frac{1}{y(y-3)}\, dy = -\frac{1}{3}\int dt. \tag{12}$$

In order to evaluate the integral on the left–hand side of (12), we decompose the integrand by means of partial fractions as

$$\frac{1}{y(y-3)} = \frac{A}{y} + \frac{B}{y-3}, \tag{13}$$

where the constants A and B are to be determined. Once A and B are determined, the integral on the left–hand side of (12) can be evaluated by virtue of (13) to obtain

$$\int \frac{1}{y(y-3)}\, dy = A\ln|y| + B\ln|y-3| + c, \tag{14}$$

for arbitrary constant c.

In order to determine A and B, multiply on both sides of the equation in (13) by $y(y-3)$ to obtain

$$1 = A(y-3) + By,$$

or

$$0y + 1 = (A+B)y - 3A. \tag{15}$$

Equating corresponding coefficients for the polynomials on the each side of (15) yields the system

$$\begin{cases} A + B &= 0 \\ -3A &= 1. \end{cases} \tag{16}$$

Solving the system in (16) yields

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{1}{3}. \tag{17}$$

Substituting the values for A and B in (17) into (14) yields the left–hand side of (12) so that, integrating both sides of (12),

$$-\frac{1}{3}\ln|y| + \frac{1}{3}\ln|y-3| = -\frac{1}{3}t + c_1, \tag{18}$$

for arbitrary constant c_1. Next, multiply on both sides of (18) by 3 and simplify to get

$$\ln\left(\frac{|y-3|}{|y|}\right) = -t + c_2, \tag{19}$$

for arbitrary constant c_2. Apply the exponential function on both sides of (19) to obtain

$$\frac{|y-3|}{|y|} = c_3\, e^{-t}, \tag{20}$$

where we have set $c_3 = e^{c_2}$. Using the continuity of y and the exponential function we get from (20) that

$$\frac{y-3}{y} = c\, e^{-t}, \tag{21}$$

for arbitrary constant c. Solving for y in (21) yields the general solution,

$$y(t) = \frac{3}{1 - c\, e^{-t}}, \tag{22}$$

for the differential equation in (11). Substituting the initial condition in (11) into (21), we get

$$c = -2. \tag{23}$$

Substituting the value of c in (22) into (22) yields a solution to the initial value problem in (11) given by

$$y(t) = \frac{3}{1 + 2\, e^{-t}}, \qquad \text{for } t \in \mathbb{R}. \tag{24}$$

A sketch of the graph of $y = y(t)$, where $y(t)$ is given in (24) is given in Figure 1 on page 5. □

4. Use partial fractions to evaluate the integral $\displaystyle\int \frac{y^3+3}{y^2-3y+2}\, dy$.

Suggestion: First divide the denominator into the numerator to obtain

$$\frac{y^3+3}{y^2-3y+2} = y + 3 + \frac{7y-3}{y^2-3y+2}. \tag{25}$$

***Solution**:* Integration on both sides of (25) yields

$$\int \frac{y^3+3}{y^2-3y+2}\, dy = \frac{1}{2}y^2 + 3y + \int \frac{7y-3}{y^2-3y+2}\, dy. \tag{26}$$

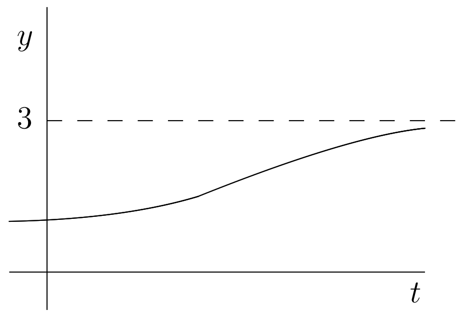

Figure 1: Sketch of solution to IVP (11)

In order to evaluate the right–most integral in (26), we first factor the denominator in the integrand to obtain

$$\frac{7y-3}{y^2-3y+2} = \frac{7y-3}{(y-1)(y-2)}. \tag{27}$$

Next, decompose the right–hand side of (27) into partial fractions to obtain

$$\frac{7y-3}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}, \tag{28}$$

where A and B are to be determined. It then follows from (27) and (28) that

$$\int \frac{7y-3}{y^2-3y+2}\,dy = A\ln|y-1| + B\ln|y-2| + c, \tag{29}$$

where c is an arbitrary constant.

In order to determine A and B, multiply on both sides of the equation in (28) by $(y-1)(y-2)$ to obtain

$$7y-3 = A(y-2) + B(y-1),$$

or

$$7y-3 = (A+B)y - 2A - B. \tag{30}$$

Equating corresponding coefficients for the polynomials on the each side of (30) yields the system

$$\begin{cases} A+B &= 7 \\ -2A-B &= -3. \end{cases} \tag{31}$$

Solving the system in (31) yields

$$A = -4 \quad \text{and} \quad B = 11. \tag{32}$$

Substituting the values for A and B in (33) into (29) yields

$$\int \frac{7y-3}{y^2-3y+2}\, dy = -4\ln|y-1| + 11\ln|y-2| + c, \tag{33}$$

for arbitrary constant c. Combining (26) and (33)

$$\int \frac{y^3+3}{y^2-3y+2}\, dy = \frac{1}{2}y^2 + 3y + -4\ln|y-1| + 11\ln|y-2| + c,$$

for arbitrary constant c. □

5. Use partial fractions to evaluate the integral $\displaystyle\int \frac{1}{1-y^2}\, dy$.

***Solution**:* First, rewrite the integral as

$$\int \frac{1}{1-y^2}\, dy = -\int \frac{1}{y^2-1}\, dy. \tag{34}$$

Factor the denominator in the integrand in the right–most integral in (34) to get

$$\frac{1}{y^2-1} = \frac{1}{(y+1)(y-1)}. \tag{35}$$

We decompose the right–hand side in (35) by means of partial fractions as

$$\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}, \tag{36}$$

where the constants A and B are to be determined. Once A and B are determined, the right–most integral in (34) can be evaluated by virtue of (35) and (36) to obtain

$$\int \frac{1}{y^2-1}\, dy = A\ln|y+1| + B\ln|y-1| + c, \tag{37}$$

for arbitrary constant c.

In order to determine A and B, multiply on both sides of the equation in (36) by $(y+1)(y-1)$ to obtain

$$1 = A(y-1) + B(y+1),$$

or

$$0y + 1 = (A + B)y + B - A. \tag{38}$$

Equating corresponding coefficients for the polynomials on the each side of (38) yields the system

$$\begin{cases} A + B &= 0 \\ B - A &= 1. \end{cases} \tag{39}$$

Solving the system in (39) yields

$$A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}. \tag{40}$$

Substituting the values for A and B in (40) into (37) yields

$$\int \frac{1}{y^2 - 1}\, dy = -\frac{1}{2} \ln|y + 1| + \frac{1}{2} \ln|y - 1| + c, \tag{41}$$

for arbitrary constant c. It then follows from (34) and (41) that

$$\int \frac{1}{1 - y^2}\, dy = \frac{1}{2} \ln|y + 1| - \frac{1}{2} \ln|y - 1| + c,$$

for arbitrary constant c, or

$$\int \frac{1}{1 - y^2}\, dy = \frac{1}{2} \ln\left(\frac{|y + 1|}{|y - 1|}\right) + c.$$

□