Solutions to Assignment #16

1. *Logistic Growth*[1]. Suppose that the growth of a certain animal population is governed by the differential equation

$$\frac{1000}{N}\frac{dN}{dt} = 100 - N, \tag{1}$$

where $N(t)$ denote the number of individuals in the population at time t.

(a) Suppose there are 200 individuals in the population at time $t = 0$. Sketch the graph of $N = N(t)$.

***Solution**:* The equation in (1) describes logistic growth in a population with intrinsic growth rate $r = 100/1000$ and carrying capacity $K = 100$. A sketch of the solution with initial population $N(0) = 200$ is shown in Figure 1. □

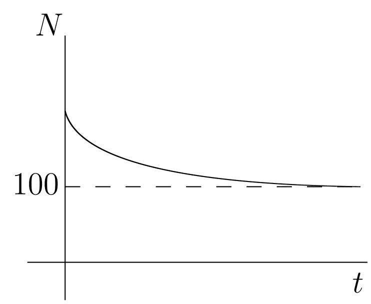

Figure 1: Sketch of Solution to (1) with $N_o = 200$

(b) Will there ever be more than 200 individuals in the population? Will there ever be fewer than 100 individuals? Explain your answer.

***Solution**:* The sketch of the solution to (1) subject to the initial condition $N(0) = 200$ shows that the population size will never be above 200 or below 100. □

2. *Spread of a viral infection*[2]. Let $I(t)$ denote the total number of people infected with a virus. Assume that $I(t)$ grows according to a logistic model. Suppose

[1] Adapted from Problem 6 on page 521 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002

[2] Adapted from Problem 7 on page 521 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002

that 10 people have the virus originally and that, in the early stages of the infection the number of infected people doubles every 3 days. It is also estimated that, in the long run 5000 people in a given area will become infected.

(a) Solve an appropriate logistic model to find a formula for computing $I(t)$, where t is the time from the initial infection measured in weeks. Sketch the graph of $I(t)$.

***Solution**:* The function I solves the logistic equation

$$\frac{dI}{dt} = rI(K - I), \tag{2}$$

where r is the intrinsic growth rate of infection and K is the limiting number of people who will become infected in the long run. Thus,

$$K \doteq 5000. \tag{3}$$

In order to estimate r, we approximate the spread of the infection with an exponential model with doubling time of 3 days or 3/7 weeks. Thus,

$$r \doteq \frac{\ln 2}{3/7} \doteq 1.6173, \tag{4}$$

in units of 1/week.

The solution to (2) subject to the initial condition $I(0) = I_o$ is given by

$$I(t) = \frac{I_o K}{I_o + (K - I_o)e^{-rt}}, \quad \text{for } t \in \mathbb{R}. \tag{5}$$

Substituting the values of $I_o = 10$, and K and r given in (3) and (4), respectively, into (5) yields the solution

$$I(t) = \frac{50000}{10 + (4990)e^{-1.6173t}}, \quad \text{for } t \in \mathbb{R}. \tag{6}$$

A sketch of the graph of the function in (6) is pictured in Figure 2. □

(b) Estimate the time when the rate of infected people begins to decrease.

***Solution**:* The rate of infection will begin to decrease when the number of infected people is half of the limiting value; namely, when

$$I(t) = 2500,$$

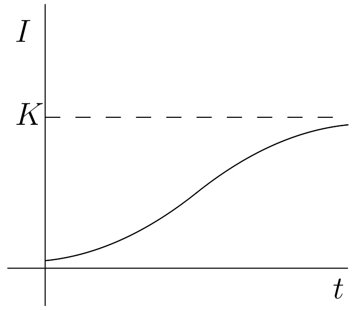

Figure 2: Sketch of function in (6)

or, according to (6), when

$$\frac{50000}{10 + (4990)e^{-1.6173t}} = 2500. \tag{7}$$

Solving the equation in (7) yields

$$t \doteq \frac{1}{1.6173}\ln(499) \doteq 3.84 \text{ weeks.}$$

Thus, the rate of infection will begin to decrease in about 3 weeks and 5 days and 21 hours. □

3. *Non–Logistic Growth*[3]. There are many classes of organisms whose birth rate is not proportional to the population size. For example, suppose that each member of the population requires a partner for reproduction, and each member relies on chance encounters for meeting a mate. Assume that the expected number of encounters is proportional to the product of numbers of female and male members in the population, and that these are equally distributed; hence, the number of encounters will be proportional to the square of the size of the population.

   Use a conservation principle to derive the population model

$$\frac{dN}{dt} = aN^2 - bN, \tag{8}$$

[3]Adapted from Problem 12 on page 39 in Braun, *Differential Equations and their Applications,* Fourth Edition, Springer–Verlag, 1993

where a and b are positive constants. Explain your reasoning.

***Solution**:* Begin with the conservation principle

$$\frac{dN}{dt} = \text{Rate of individuals in} - \text{Rate of individuals out}. \tag{9}$$

In this case we have

$$\text{Rate of individuals in} = aN^2, \tag{10}$$

and

$$\text{Rate of individuals out} bN, \tag{11}$$

where a and b are positive constants of proportionality. The equation in (8) follows from (9) after substituting (10) and (11). □

4. For the equation in (8),

(a) find the values of N for which the population size is not changing;

***Solution**:* Rewrite the equation in (8) as

$$\frac{dN}{dt} = aN\left(N - \frac{b}{a}\right). \tag{12}$$

We see from (12) that $\dfrac{dN}{dt} = 0$ when $N = 0$ or $N = \dfrac{b}{a}$. □

(b) find the range of positive values of N for which the population size is increasing, and those for which it is decreasing;

***Solution**:* We see from (12) that $\dfrac{dN}{dt} > 0$ for $N > \dfrac{b}{a}$, and $\dfrac{dN}{dt} < 0$ for $N < \dfrac{b}{a}$. This, the population size increases for $N > \dfrac{b}{a}$, and decreases for $N < \dfrac{b}{a}$. □

(c) find ranges of positive values of N for which the graph of $N = N(t)$ is concave up, and those for which it is concave down;

***Solution**:* Differentiate on both sides of (8) with respect to t to obtain

$$\frac{d^2N}{dt^2} = 2aN\frac{dN}{dt} - b\frac{dN}{dt}, \tag{13}$$

where we have applied the Chain Rule. The equation in (13) can be rewritten as

$$\frac{d^2N}{dt^2} = 2a\left(N - \frac{b}{2a}\right)\frac{dN}{dt}. \tag{14}$$

Substituting the expression for $\dfrac{dN}{dt}$ in (12) into (14) then yields

$$\frac{d^2N}{dt^2} = 2a^2N\left(N - \frac{b}{2a}\right)\left(N - \frac{b}{a}\right). \tag{15}$$

In view of (15) we see that, for positive values of N, the sign of $\dfrac{d^2N}{dt^2}$ is determined by the signs of the two right–most factors in (15). The signs of these two factors are displayed in Table 1. The concavity of of the graph

	0 to $b/2a$	$b/2a$ to b/a	$> b/a$
$N - \frac{b}{2a}$	$-$	$+$	$+$
$N - \frac{b}{a}$	$-$	$-$	$+$
$N''(t)$	$+$	$-$	$+$
graph of $N(t)$	concave–up	concave–down	concave–up

Table 1: Concavity of the graph of $N = N(t)$

of $N = N(t)$ is also displayed in Table 1. From that table we get that the graph of $N = N(t)$ is concave up for

$$0 < N < \frac{b}{2a} \quad \text{or} \quad N > \frac{b}{a},$$

and concave down for

$$\frac{b}{2a} < N < \frac{b}{a}.$$

□

(d) Sketch possible solutions.

***Solution**:* Putting together the information on concavity in Table 1 and the fact that $N(t)$ increases for $N > b/a$ and decreases for $0 < N < b/a$,

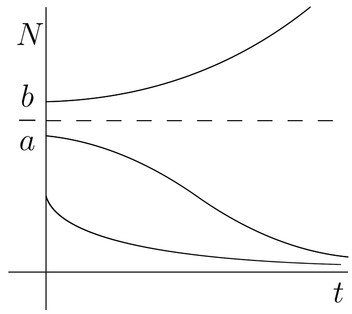

Figure 3: Possible Solutions to Logistic equation

we obtain the sketches of possible solutions to the equation in (8) displayed in Figure 3.

□

5. For the equation in (8),

   (a) use separation of variables and partial fractions to find a solution satisfying the initial condition $N(0) = N_o$, for $N_o > 0$.

   ***Solution**:* Separate variable in the equation in (12) to obtain

$$\int \frac{1}{N(N - b/a)}\, dN = \int a\ dt. \tag{16}$$

Use partial fractions in the integrand on the left–hand side to (16) and integrate on the right–hand side to get to get

$$\frac{a}{b}\int \left\{-\frac{1}{N} + \frac{1}{N - b/a}\right\} dN = at + c_1, \tag{17}$$

for some constant c_1. Evaluate the integral on the left–hand side of (17) and simplify to get

$$\ln\left(\frac{|N - b/a|}{|N|}\right) = bt + c_2, \tag{18}$$

for some constant c_2. Next, take the exponential function on both sides of (18) to get

$$\frac{|N - b/a|}{|N|} = c_3\ e^{bt}, \tag{19}$$

where we have set $c_3 = e^{c_2}$.

Using the continuity of N and of the exponential function we deduce from (19) that

$$\frac{N - b/a}{N} = c\ e^{a^2 t/b}, \tag{20}$$

for some constant c. The equation in (20) can now be solved for N as a function of t to get

$$N(t) = \frac{b/a}{1 - c\ e^{bt}}. \tag{21}$$

Next, use the initial condition $N(0) = N_o$ to obtain from (20) that

$$c = \frac{N_o - b/a}{N_o}. \tag{22}$$

Substituting the value of c in (22) into (21) yields

$$N(t) = \frac{N_o b/a}{N_o + (b/a - N_o)\ e^{bt}}. \tag{23}$$

□

(b) What happens to $N(t)$ as $t \to \infty$ if $N_o > b/a$? What happens if $N_o < b/a$? Why is b/a called a threshold value?

***Solution**:* We first consider the case in which $0 < N_o < b/a$. In this case, the function in (23) is defined for all values of t and

$$\lim_{t \to \infty} N(t) = 0,$$

since $b > 0$.

On the other hand, is $N_o > b/a$, then the function in (23) ceases to exist when

$$(N_o - b/a)\ e^{bt} = N_o.$$

As t approaches that time, $N(t) \to \infty$. Thus, depending on whether $N_o < b/a$ or $N_o > b/a$, the population will eventually go extinct or it will have unlimited growth in a finite time. Thus, b/a is the threshold population value which determines growth or extinction. □