Solutions to Assignment #1

1. Let f denote a differentiable on some, non–empty, open interval, I. Assume that $f'(t) = 0$ for all t in the interval I. Use the Fundamental Theorem of Calculus to show that f must be constant on I.

   Suggestion: Let $a \in I$ and use the Fundamental Theorem of Calculus to show that $f(t) = f(a)$ for all t in I.

   ***Solution**:* Applying the Fundamental Theorem of Calculus we have that
   $$f(t) - f(a) = \int_a^t f'(\tau)\, d\tau = 0,$$
   since $f'(\tau) = 0$ for all $\tau \in I$. Consequently,
   $$f(t) = f(a), \qquad \text{for all } t \in I,$$
   which shows that f is constant on I. □

2. Assume that $u = u(t)$ and $v = v(t)$ are differentiable functions of a single variable, t, defined on a non–empty, open interval, I. Suppose that $v(t) > 0$ for all t in I and that
   $$u'(t)v(t) - u(t)v'(t) = 0, \qquad \text{for all } t \text{ in } I.$$
   Show that u must be a constant multiple of v; that is, there is a constant, c, such that
   $$u(t) = c\; v(t), \qquad \text{for all } t \text{ in } I.$$
   Suggestion: Use the quotient rule to compute the derivative, $\left(\dfrac{u}{v}\right)'$, of $\dfrac{u}{v}$.

   ***Solution**:* Apply the quotient rule to compute
   $$\left(\frac{u}{v}\right)' = \frac{vu' - u'v}{v^2} = 0, \qquad \text{on } I.$$
   Hence, by the result from Problem 1, $\dfrac{u}{v}$ is constant on I, so that
   $$\frac{u}{v} = c,$$
   which implies that $u = cv$, which was to be shown. □

A Conservation Principle for a One–Compartment Model. Suppose you are tracking the amount, $Q(t)$, of a substance in some predefined space or region, known as a *compartment,* at time t. (A compartment could represent, for instance, the bloodstream in the body of a patient, and $Q(t)$ the amount of a drug present in the bloodstream at time t). If we know, or can model, the rates at which the substance enters or leaves the compartment, then the rate of the change of the substance in the compartment is determined by the differential equation:

$$\frac{dQ}{dt} = \text{Rate of substance in} - \text{Rate of substance out}, \tag{1}$$

where we are assuming that Q is a differentiable function of time.

3. Assume that the rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time.

   Use the conservation principle (1) to write down a differential equation for the quantity, Q, of the drug in the blood at time, t, in hours.

   ***Solution**:* Let $Q = Q(t)$ denote the amount of drug present in the blood at time t. We apply the conservation principle in (1). In this case
   $$\text{Rate of substance in} = 0$$
   and
   $$\text{Rate of substance out} = kQ,$$
   where k is a constant of proportionality. Hence,
   $$\frac{dQ}{dt} = -kQ.$$
   $\square$

4. A patient is given the drug theophylline intravenously at a constant rate of 43.2 mg/hour to relieve acute asthma. You can imagine the drug as entering a compartment of volume $35,000$ ml. (This is an estimate of the volume of the part of the body through which the drug circulates.) The rate at which the drug leaves the patient is proportional to the quantity there, with proportionality constant 0.082.

   Write a differential equation for the quantity, $Q = Q(t)$, in milligrams, of the drug in the body at time t hours.

Solution: Let $Q = Q(t)$ denote the amount of theophylline present in the patient's body at time t. We apply the conservation principle in (1). In this case

$$\text{Rate of substance in} = 43.2$$

and

$$\text{Rate of substance out} = kQ,$$

where $k = 0.082$. Hence,

$$\frac{dQ}{dt} = 43.2 - (0.082)Q,$$

in mg/hour. □

5. When people smoke, carbon monoxide is released into the air. Suppose that in a room of volume 60 m^3, air containing 5% carbon monoxide is introduced at a rate of 0.002 m^3/min. (This means that 5% of the volume of incoming air is carbon monoxide). Assume that the carbon monoxide mixes immediately with the air and the mixture leaves the room at the same rate as it enters.

   (a) Let $Q = Q(t)$ denote the volume (in cubic meters) of carbon monoxide in the room at any time t in minutes. Use the conservation principle (1) to write down a differential equation for Q.

   Solution: Imagine the room as a compartment of, fixed volume, V. In this case, $V = 60$ m^3. Air flows into the room at rate, F, of 0.002 cubic meters per minute. The air that flows into the room has a concentration, c_i, of carbon monoxide, where $c_i = 5\%$ (the concentration here is measured in percent volume). Let $Q(t)$ denote the amount of carbon monoxide present in the room at time t. The conservation principle in (1) in this case takes the form

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},$$

   where

$$\text{Rate of } Q \text{ in} = c_i F,$$

   and

$$\text{Rate of } Q \text{ out} = c(t)F,$$

   where $c(t) = \dfrac{Q(t)}{V}$ is the concentration of carbon monoxide in the room at time t. Here we are assuming that the volume, V, of air

in the room is fixed, so that the rate of flow of air into the room is the same as the rate of flow out of the room.
We then have that

$$\frac{dQ}{dt} = c_i F - \frac{Q(t)}{V} F.$$

Putting in the values of F, V and c_i we obtain

$$\frac{dQ}{dt} = 10^{-4} - \frac{1}{3} \times 10^{-4} Q, \tag{2}$$

in units of cubic meters per minute. □

(b) Based on your answer to part (a), give a differential equation satisfied by the concentration, $c(t)$, of carbon monoxide in the room (in percent volume) at any time t in minutes.

Solution: Divide the differential equation in (2) by the volume $V = 60$ m^3 to obtain

$$\frac{dc}{dt} = \frac{1}{6} \times 10^{-5} - \frac{1}{3} \times 10^{-4} c.$$

where we have used the expression

$$c = \frac{Q}{V}.$$

□