Assignment #2

Due on Wednesday, September 14, 2016

Read Section 3.1, *Preliminary Analysis of the Logistic Equation,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.2, *Mathematical Questions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let $N(t)$ denote the size of a bacterial population in culture at time t. $N(t)$ can be measured by weight (e.g., grams), or by concentration via optical density measurements. Assume that $N = N(t)$ is twice differentiable and that it satisfies the following differential equation:

$$\frac{dN}{dt} = 1.24N - 3.60N^2, \tag{1}$$

where $N = N(t)$ measures the concentration of bacteria obtained via optical density measurements.

Using the information provided by the differential equation in (1),

(a) find the values of N for which the population size is not changing; that is the values of N for which $\dfrac{dN}{dt} = 0$;

(b) find the range of positive values of N for which the population size is increasing; that is the values of N for which $\dfrac{dN}{dt} > 0$;

(c) find the range of positive values of N for which the population size is decreasing; that is the values of N for which $\dfrac{dN}{dt} < 0$.

2. Use the differential equation in (1) and the Chain Rule to obtain an expression for the second derivative of N with respect to t, $\dfrac{d^2N}{dt^2}$. Put your answer in the form

$$\frac{d^2N}{dt^2} = g(N), \tag{2}$$

where g is a function of a single variable.

3. Based on your answer to Problem 2 in the form of equation (2),

   (a) find the values of N for which the graph of $N = N(t)$ (that is, graph of N as a function of t in the tN–plane), might have an inflection point; that is, find the values of N for which $\dfrac{d^2N}{dt^2} = 0$;

   (b) find the range of positive values of N for which the graph of $N = N(t)$ is concave up; that is the values of N for which $\dfrac{d^2N}{dt^2} > 0$;

   (c) find the range of positive values of N for which the graph of $N = N(t)$ is concave down; that is the values of N for which $\dfrac{d^2N}{dt^2} < 0$.

4. Suppose that $N = N(t)$ is a solution to the differential equation in (1). Use the qualitative information about the graph of $N = N(t)$ obtained in Problems 2 and 3 to sketch possible graphs of N for $N \geqslant 0$.

   Based on your sketches, explain what the population model in (1) seems to be predicting.

5. Analysis of certain one–compartment dilution model yields the differential equation

$$\frac{dQ}{dt} = a\left(1 - \frac{Q}{L}\right), \qquad (3)$$

   for positive constants a and L.

   Assume that the differential equation in (3) has a solution, $Q = Q(t)$, which is twice–differentiable.

   (a) Determine the value, or values, of Q for which $\dfrac{dQ}{dt} = 0$.

   (b) Find a range of positive values of Q on which $Q(t)$ is increasing, and those values of Q for which $Q(t)$ is decreasing.

   (c) Determine values of Q on which the graph of $Q = Q(t)$ is concave up, and those on which it is concave down.

   (d) Use the qualitative information obtained in parts (b) and (c) to sketch possible graphs of a solution, $Q = Q(t)$, of the differential equation in (3), for positive values of Q.

   Based on your sketches, explain what the equation in (3) seems to be predicting.