Solutions to Assignment #3

1. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = t^2 \\ y(0) = 2. \end{cases}$$

***Solution**:* Compute

$$y(t) = 2 + \int_0^t \tau^2 \, d\tau = 2 + \frac{t^3}{3}, \qquad \text{for all } t \in \mathbb{R}.$$

□

2. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \sqrt{t} \\ y(1) = 0. \end{cases}$$

***Solution**:* Compute

$$\begin{aligned} y(t) &= \int_1^t \sqrt{\tau} \, d\tau \\ &= \left[\frac{\tau^{3/2}}{3/2} \right]_1^t \\ &= \frac{2}{3} t^{3/2} - \frac{2}{3}, \end{aligned}$$

for $t \geqslant 0$. □

3. Let $y = y(t)$ denote the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{1 + t^4} \\ y(0) = 0. \end{cases}$$

(a) Use

$$y(t) = y_o + \int_{t_o}^{t} f(\tau)\, d\tau, \qquad \text{for all } t \in I,$$

to write down a formula for computing $y(t)$.

Solution:

$$y(t) = \int_0^t \frac{1}{1+\tau^4}\, d\tau, \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

□

(b) Compute $y'(t)$ and $y''(t)$.

Solution: It follows from (1) and the Fundamental Theorem of Calculus that

$$y'(t) = \frac{1}{1+t^4}, \qquad \text{for all } t \in \mathbb{R}. \tag{2}$$

Differentiating $y'(t)$ with respect to t we obtain, by the Chain Rule, that

$$y''(t) = -\frac{4t^3}{(1+t^4)^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{3}$$

□

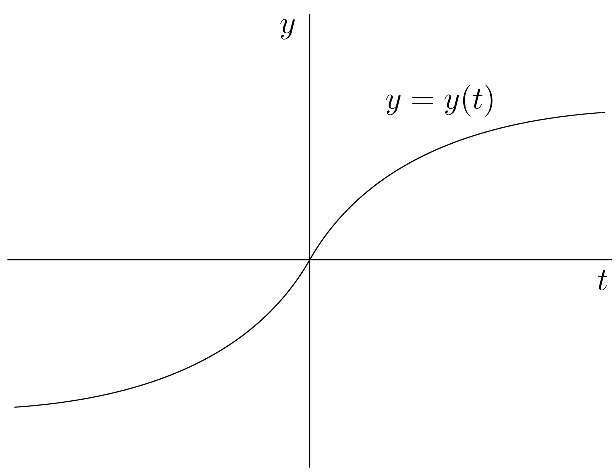

Figure 1: Sketch of graph of $y = y(t)$

(c) Determine intervals on which (i) $y(t)$ increases, (ii) $y(t)$ decreases, (iii) the graph of $y = y(t)$ is concave up, and (iv) the graph of $y = y(t)$ is concave down.

Solution: It follows from (2) that $y'(t) > 0$ for all values of t, so that $y(t)$ increases as t increases for all $t \in \mathbb{R}$.
Using (3) we see that $y''(t) < 0$ for positive values of t and $y''(t) > 0$ for negative values of t. Thus, the graph of $y = y(t)$ is concave down for $t > 0$ and concave up for $t < 0$. □

(d) Sketch the graph of $y = y(t)$.

Solution: Putting together the qualitative information obtained in part (c), we obtain the graph shown in Figure 1. Observe that the graph has an inflection point at $(0, 0)$. □

4. Let $f(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$

   (a) Explain why f is continuous at 0.

   Solution: Using the limit

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

   we see that $\lim_{t\to 0} f(t) = f(0)$, so that f is continuous at 0. □

   (b) Use the fundamental Theorem of Calculus to write an expression for the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t) \\ y(0) = 0. \end{cases}$$

   Solution: $y(t) = \displaystyle\int_0^t \frac{\sin \tau}{\tau}\, d\tau$ for all $t \in \mathbb{R}$. □

5. Define

$$F(t) = \int_0^{t^2} \frac{\sin(\tau)}{\tau}\, d\tau, \qquad \text{for } t \in \mathbb{R}.$$

   Use the Fundamental Theorem of Calculus and the Chain Rule to compute $F'(t)$.

Solution: Put $G(t) = \int_0^t \frac{\sin(\tau)}{\tau}\, d\tau$, for $t \in \mathbb{R}$. Then,

$$F(t) = G(t^2), \quad \text{for all } t \in \mathbb{R}.$$

By the Chain Rule we have that

$$F'(t) = 2t\, G'(t^2), \quad \text{for all } t \in \mathbb{R}, \tag{4}$$

where, by virtue of the Fundamental Theorem of Calculus,

$$G'(t) = \frac{\sin(t)}{t}, \quad \text{for all } t \in \mathbb{R}. \tag{5}$$

Combining equations (4) and (5) we obtain that

$$F'(t) = 2t\, \frac{\sin(t^2)}{t^2}, \quad \text{for } t \in \mathbb{R},$$

or

$$F'(t) = 2\, \frac{\sin(t^2)}{t}, \quad \text{for } t \in \mathbb{R}.$$

□