Solutions to Assignment #5

Background and Definitions

The natural logarithm function, $\ln\colon (0,\infty)\to\mathbb{R}$, is the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{t}; \\ y(1) = 0, \end{cases}$$

for $t > 0$, so that

$$\ln(t) = \int_1^t \frac{1}{\tau}\, d\tau, \qquad \text{for all } t > 0.$$

Using this definition, we derived the follow properties of the natural logarithm function in class.

(i) $\ln(1) = 0$;

(ii) $\ln\colon (0,\infty)\to\mathbb{R}$ is differentiable and $\ln'(t) = \dfrac{1}{t}$, for all $t > 0$;

(iii) $\ln(ab) = \ln a + \ln b$ for all $a, b > 0$;

(iv) $\ln(b^p) = p\ \ln b$ for all $b > 0$ and $p \in \mathbb{R}$.

1. Derive the following additional properties of the natural logarithm function.

   (a) $\ln\left(\dfrac{1}{b}\right) = -\ln b$, for $b > 0$.

   ***Solution**:* Write $\dfrac{1}{b} = b^{-1}$ so that, applying property (iv),

$$\begin{aligned} \ln\left(\frac{1}{b}\right) &= \ln(b^{-1}) \\ &= (-1)\ln(b) \\ &= -\ln b, \end{aligned}$$

   which was to be shown. □

(b) $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b,\;$ for $a, b > 0$.

***Solution**:* Write $\dfrac{a}{b} = a \cdot \dfrac{a}{b}$ so that, applying property (iii)

$$\begin{aligned} \ln\left(\frac{a}{b}\right) &= \ln\left(a \cdot \frac{1}{b}\right) \\ &= \ln(a) + \ln\left(\frac{1}{b}\right), \end{aligned}$$

and so, using the result of part (a) in this problem,

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln b,$$

which was to be shown. □

2. Let $f(t) = \ln\sqrt{1+t^2}$ for all $t \in \mathbb{R}$.

(a) Compute $f'(t)$ and $f''(t)$.

***Solution**:* Write $f(x) = \dfrac{1}{2}\ln(1+t^2)$ and compute

$$f'(t) = \frac{1}{2}\ln'(1+t^2)\cdot\frac{d}{dt}[1+t^2],$$

where we have applied the Chain Rule, so that

$$f'(t) = \frac{1}{2}\cdot\frac{1}{1+t^2}\cdot(2t),$$

or

$$f'(t) = \frac{t}{1+t^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

Next, differentiate with respect to t the expression for $f'(t)$ in (1), applying the quotient rule, to obtain

$$\begin{aligned} f''(t) &= \frac{d}{dt}\left[\frac{t}{1+t^2}\right] \\ &= \frac{1+t^2 - t(2t)}{(1+t^2)^2}, \end{aligned}$$

so that

$$f''(t) = \frac{1-t^2}{(1+t^2)^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{2}$$

□

(b) Determine the intervals on the t–axis for which f is increasing or decreasing, and all local extrema; the values of t for which the graph of $y = f(t)$ is concave up, and those for which the graph is concave down; and all the inflection points of the graph of $y = f(t)$. Sketch the graph of $y = f(t)$.

***Solution**:* From the expression for $f'(t)$ in (1) we obtain that $f'(t) > 0$ for $t > 0$ and $f'(t) < 0$ for $t < 0$. This, $f(t)$ increases for $t > 0$ and decreases for $t < 0$. We conclude from this information that $f(t)$ is a minimum when $t = 0$; so that

$$\min f = f(0) = \ln \sqrt{1} = 0.$$

Next, write the expression for $f''(t)$ in (2) as

$$f''(t) = \frac{(1+t)(1-t)}{(1+t^2)^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{3}$$

We see from the expression for $f''(t)$ in (3) that the sign of $f''(t)$ is determined by the signs of the factors, $1+t$ and $1-t$, in the numerator on for the right–hand side in (3). The signs of these factors are displayed in Table 1. The concavity of the graph of $y = f(t)$ is also shown in Table

	$t < -1$	$-1 < t < 1$	$t > 1$
$1+t$:	$-$	$+$	$+$
$1-t$:	$+$	$+$	$-$
$f''(t)$:	$-$	$+$	$-$
Concavity:	down	up	down

Table 1: Concavity of the graph of $y = f(t)$

1. From the information in the table, we also conclude that the graph of $y = f(t)$ has inflection points at the points

$$(-1, \ln\sqrt{2}) \quad \text{and} \quad (-1, \ln\sqrt{2}).$$

A sketch of the graph of $y = f(t)$ is shown in Figure 1.

□

3. Let $f(t) = t \ln t$ for $t > 0$.

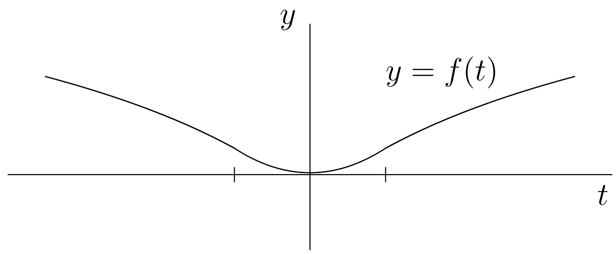

Figure 1: Sketch of graph of $y = f(t)$

(a) Compute $f'(t)$ and $f''(t)$.

***Solution**:* First, apply the product rule to compute

$$f'(t) = \ln t + t \cdot \frac{1}{t},$$

so that

$$f'(t) = \ln t + 1, \quad \text{for } t > 0. \tag{4}$$

Differentiating the expression for $f'(t)$ in (6) with respect to t yields

$$f''(t) = \frac{1}{t}, \quad \text{for } t > 0. \tag{5}$$

□

(b) Determine the intervals on the t–axis for which f is increasing or decreasing, and all local extrema; the values of t for which the graph of $y = f(t)$ is concave up, and those for which the graph is concave down; and all the inflection points of the graph of $y = f(t)$. Sketch the graph of $y = f(t)$.

***Solution**:* It follows from (7) that the graph of $y = f(t)$ is concave up for all $t > 0$.

From (6) we see that $f'(t) = 0$ when $t = e^{-1}$. Also, $f'(t) > 0$ for t with

$$\ln t > -1 = \ln(e^{-1}),$$

which implies that $t > e^{-1}$ (since ln is an increasing function). Hence, $f(t)$ increases for $t > e^{-1}$. Similarly, $f(t)$ decreases for $0 < t < e^{-1}$. We then have that $f(t)$ has a minimum $t = e^{-1}$.

A sketch of the graph of $y = f(t)$ is shown in Figure 2. □

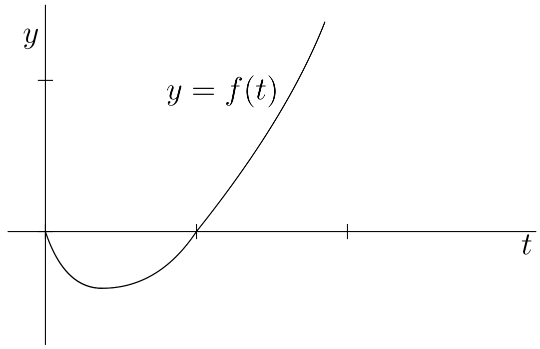

Figure 2: Sketch of graph of $y = f(t)$ in Problem 3

4. Evaluate the indefinite integral

$$\int \frac{1}{t + \sqrt{t}}\, dt \tag{6}$$

by making the change of variables $u = \sqrt{t}$.

***Solution**:* Let $u = \sqrt{t}$. Then, $du = \dfrac{1}{2\sqrt{t}}\, dt$, so that $dt = 2u\, du$. Substituting into the integral in (6) we obtain that

$$\begin{aligned} \int \frac{1}{t + \sqrt{t}}\, dt &= \int \frac{2u}{u^2 + u}\, du \\ &= 2\int \frac{1}{u+1}\, du. \end{aligned} \tag{7}$$

Making a further change of variables, $v = u + 1$, in the last integral in (7) we obtain

$$\begin{aligned} \int \frac{1}{t + \sqrt{t}}\, dt &= 2\int \frac{1}{v}\, dv \\ &= 2\ln|v| + c. \end{aligned} \tag{8}$$

Substituting back in terms of u and t, we obtain from (8) that

$$\int \frac{1}{t + \sqrt{t}}\, dt = 2\ln(1 + \sqrt{t}) + c, \qquad \text{for } t > 0.$$

$\square$

5. Define $g(t) = t\ln t - t$ for all $t > 0$. Compute $g'(t)$ and use your result in order to obtain a formula for evaluating the indefinite integral

$$\int \ln u \; du.$$

***Solution**:* Applying the product rule we obtain

$$g'(t) = \ln t + t \cdot \frac{1}{t} - 1 = \ln t, \qquad \text{for all } t > 0.$$

Consequently,

$$\int \ln u \; du = u\ln u - u + c.$$

□