Solutions to Assignment #6

1. Show that $\displaystyle\int_1^{2.5} \frac{1}{\tau}\, d\tau < 1$ by comparing the area under the graph of $y = 1/\tau$ from $\tau = 1$ to $\tau = 2.5$ with the sum of the areas of circumscribed rectangles of width 0.25.

   Use this result to conclude that $2.5 < e$.

   ***Solution**:* The circumscribed rectangles are shown in Figure 1. By comparing

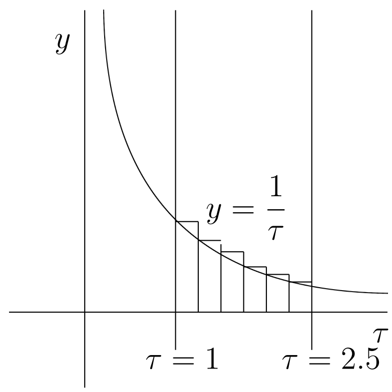

Figure 1: Circumscribed Rectangles for Problem 1

the area of the region under the graph of $t = 1/\tau$ above the t–axis and between the lines $\tau = 1$ and $\tau = 2.5$ (in other words, $\ln(2.5)$) with the area of the circumscribed rectangles in Figure 1, we see that

$$\ln(2.5) < \text{area of circumscribed rectangles}, \tag{1}$$

where

$$\begin{aligned} \text{area of circumscribed rectangles} &= \frac{1}{4}\left[1 + \frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4} + \frac{1}{2} + \frac{1}{9/4}\right] \\ &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \\ &= \frac{2509}{2520}. \end{aligned} \tag{2}$$

It follows from (1) and the calculations in (3) that

$$\ln(2.5) < 1,$$

so that

$$\ln(2.5) < \ln e, \tag{3}$$

Next, use the fact that $\ln(t)$ is an increasing function of t to conclude from (3) that

$$2.5 < e,$$

which was to be shown. □

2. In class and in the lecture notes we showed that $2 < e < 3$. Show that

$$\int_1^{2.875} \frac{1}{\tau}\, d\tau > 1$$

by comparing the area under the graph of $y = 1/\tau$ from $\tau = 1$ to $\tau = 2.875$ with the areas of inscribed rectangles of width 0.125. Use the result of this problem and Problem 1 to conclude that $2.5 < e < 2.875$.

Solution: The inscribed rectangles are shown in Figure 2. The area of the

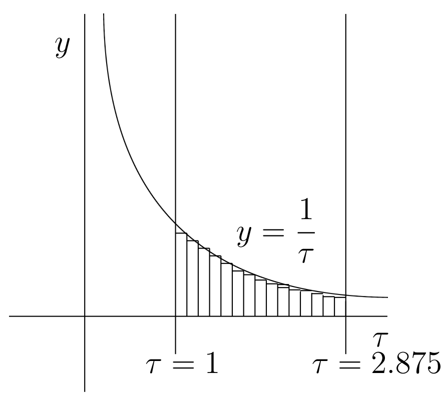

Figure 2: Inscribed Rectangles for Problem 2

inscribed rectangles shown in Figure 2 is an underestimate for $\ln(2.875)$ so that

$$\text{area of inscribed rectangles} < \ln(2.875), \tag{4}$$

where

$$\begin{aligned}\text{area of inscribed rectangles} &= \frac{1}{8}\left[\frac{1}{9/8} + \frac{1}{10/8} + \cdots + \frac{1}{23/8}\right]\\ &\approx 1.01643,\end{aligned}$$

so that

$$\text{area of inscribed rectangles} \;>\; 1. \tag{5}$$

Combining (4) and (5) we see that

$$1 < \ln(2.875),$$

or

$$\ln e < \ln(2.875). \tag{6}$$

Thus, since $\ln(t)$ is a strictly increasing function of t, it follows from (6) that

$$e < 2.875.$$

Combining this estimate with the result of Problem 1, we can say that

$$2.5 < e < 2.875.$$

□

3. **(Base 10 Logarithm Function, or Common Logarithm)**. We say that y is the logarithm to base 10 of t if $10^y = t$. We write $y = \log t$. Thus,

$$y = \log t \quad \text{if and only if} \quad 10^y = t.$$

Solve the following equations for x using common logarithms.

(i) $2^x = 10$; (ii) $e^x = 10$; (iii) $10^x = e$; and (iv) $b^x = a$,
where a and b are positive real numbers

***Solution**:*

(i) Apply the common logarithm function on both sides of the equation

$$2^x = 10$$

to obtain

$$\log(2^x) = \log(10),$$

or

$$x \log(2) = 1,$$

which yields the solution $x = \dfrac{1}{\log(2)}$.

(ii) Proceed as in (i) to obtain that $x = \dfrac{1}{\log(e)}$.

(iii) Here, we need to assume that $b \neq 1$; otherwise, the equation has no solutions, unless $a = 1$. Apply the common logarithm function on both sides of the equation

$$10^x = e$$

to obtain

$$x = \log(e).$$

(iv) Apply the common logarithm function on both sides of the equation

$$b^x = a$$

to obtain

$$\log(b^x) = \log(a),$$

or

$$x\log(b) = \log(a),$$

which yields the solution $x = \dfrac{\log(a)}{\log(b)}$.

□

4. Suppose that $y = \log t$, for some positive real number t. Show that $y = \dfrac{\ln t}{\ln 10}$.

Solution: From $y = \log(t)$ we obtain that

$$10^y = t. \tag{7}$$

Apply the natural logarithm function to both sides of the equation in (7) to obtain

$$\ln(10^y) = \ln(t),$$

or

$$y\ln(10) = \ln(t),$$

from which we get that $y = \dfrac{\ln(t)}{\ln(10)}$. □

5. Derive the formula $\ln t = \dfrac{\log t}{\log e}$, for all $t > 0$.

Solution: Set $y = \ln(t)$; then,

$$e^y = t. \tag{8}$$

Apply the common logarithm function to both sides of the equation in (8) to obtain

$$\log(e^y) = \log(t),$$

or

$$y \log(e) = \log(t),$$

from which we get that $y = \dfrac{\log(t)}{\log(e)}$, or $\ln t = \dfrac{\log(t)}{\log(e)}$, since $y = \ln t$. $\square$