Solutions to Assignment #7

Background and Definitions

The exponential function, $\exp\colon \mathbb{R} \to (0, \infty)$, is the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y; \\ y(0) = 1, \end{cases} \tag{1}$$

for $t \in \mathbb{R}$. We therefore have that

$$\exp'(t) = \exp(t), \qquad \text{for all } t \in \mathbb{R}, \quad \exp(0) = 1,$$

and exp is the only solution to the problem in (1).

1. Show that $\exp(a - b) = \dfrac{\exp(a)}{\exp(b)}$ for all $a, b \in \mathbb{R}$.

   Solution: Write $\exp(a) = \exp(a - b + b)$ so that

   $$\exp(a) = \exp(a - b) \cdot \exp(b). \tag{2}$$

   Solving for $\exp(a - b)$ in (2) yields the result. □

2. Let r and y_o denote real numbers and put $g(t) = y_o \exp(rt)$ for all $t \in \mathbb{R}$. Show that $y = g(t)$ is the unique solution to the initial value problem

   $$\begin{cases} \dfrac{dy}{dt} = ry; \\ y(0) = y_o, \end{cases} \tag{3}$$

   by considering the function

   $$w(t) = \frac{v(t)}{\exp(rt)}, \qquad \text{for all } t \in \mathbb{R},$$

   where $v(t)$ is any solution to the initial value problem in (3).

   Solution: Let $v = v(t)$ be any solution to the initial value problem in (3); then,

   $$v'(t) = rv(t), \qquad \text{for all } t \in \mathbb{R}, \tag{4}$$

   and

   $$v(0) = y_o. \tag{5}$$

Define the function

$$w(t) = \frac{v(t)}{\exp(rt)}, \qquad \text{for all } t \in \mathbb{R}. \tag{6}$$

Differentiating the function, w, defined in (6) we obtain, by the quotient rule, that

$$w'(t) = \frac{\exp(rt)v'(t) - v(t)[\exp(rt)]'}{[\exp(rt)]^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{7}$$

Applying the Chain Rule and using (4), we obtain from (7) that

$$w'(t) = \frac{r\exp(rt)v(t) - v(t)\exp'(rt)[rt]'}{[\exp(rt)]^2}, \qquad \text{for all } t \in \mathbb{R},$$

or

$$w'(t) = \frac{r\exp(rt)v(t) - v(t)r\exp(rt)}{[\exp(rt)]^2} = 0, \qquad \text{for all } t \in \mathbb{R}. \tag{8}$$

It follows from (8) that

$$w(t) = c, \qquad \text{for all } t \in \mathbb{R}, \tag{9}$$

where c is a constant (see your result in Problem 1 of Assignment 1).

In order to find out the value of c in (9), use (9) and (6) to evaluate

$$c = \frac{v(0)}{\exp(0)} = y_o, \tag{10}$$

where we have also used (5).

Next, use (6), (9) and (10) to obtain

$$\frac{v(t)}{\exp(rt)} = y_o, \qquad \text{for all } t \in \mathbb{R},$$

from which we get

$$v(t) = y_o\exp(rt), \qquad \text{for all } t \in \mathbb{R},$$

so that any solution of (3) must in fact be equal to $y(t) = y_o\exp(rt)$ for all $t \in \mathbb{R}$. $\square$

3. Show that

$$\lim_{t\to+\infty} \exp(-t) = 0.$$

***Solution**:* Use the result of Problem 1 to write $\exp(-t) = \exp(0 - t)$ as

$$\exp(-t) = \frac{\exp(0)}{\exp(t)} = \frac{1}{\exp(t)}, \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

It follows from (11), and the fact that

$$\lim_{t\to+\infty} \exp(t) = +\infty,$$

that

$$\lim_{t\to+\infty} \exp(-t) = 0.$$

□

4. Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = 1 - \exp(-t), \quad \text{for all } t \in \mathbb{R}.$$

(a) Compute $f'(t)$ and $f''(t)$.

***Solution**:* Apply the Chain Rule to compute

$$f'(t) = -\exp'(-t) \cdot \frac{d}{dt}[-t],$$

from which we get

$$f'(t) = \exp(-t), \quad \text{for all } t \in \mathbb{R}. \tag{12}$$

Differentiate the expression of $f'(t)$ in (12) with respect to t, applying the Chain Rule, to obtain

$$f''(t) = -\exp(-t), \quad \text{for all } t \in \mathbb{R}. \tag{13}$$

□

(b) Determine the intervals on the t–axis for which f is increasing or decreasing, and all local extrema; the values of t for which the graph of $y = f(t)$ is concave up, and those for which the graph is concave down; and all the inflection points of the graph of $y = f(t)$. Sketch the graph of $y = f(t)$.

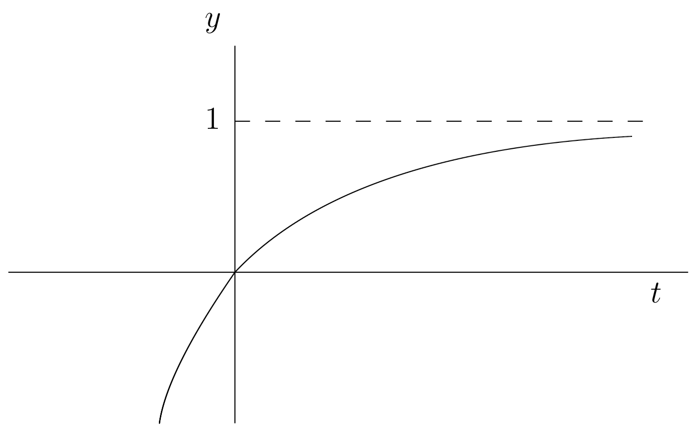

Figure 1: Sketch of graph of $y = f(t)$

***Solution**:* Since the exponential function is strictly positive, it follows from (12) that $f'(t) > 0$ for all $t \in \mathbb{R}$; hence, $f(t)$ is strictly increasing for all values of t. By the same token, we obtain from (13) that $f''(t) < 0$ for all values of t; so that the graph of $y = f(t)$ is concave down for all values of t. We also conclude that f has no extrema, and the graph of $y = f(t)$ has no inflection points. A sketch of the graph of $y = f(t)$ is shown in Figure 1. In the sketch in Figure 1, we have also taken into account the fact that

$$\lim_{t \to +\infty} [1 - \exp(-t)] = 1,$$

as a consequence of the result in Problem 3; so that the line $y = 1$ is an asymptote to the graph of $y = f(t)$. Note also

$$f(t) = 1 - \exp(-t) < 1, \quad \text{for all } t \in \mathbb{R},$$

since $\exp(-t) > 0$ for all $t \in \mathbb{R}$. Finally, we have also used the fact that

$$f(0) = 1 - \exp(0) = 1 - 1 = 0,$$

so that the graph of $y = f(t)$ goes through $(0, 0)$. □

5. Let b denote a positive real number. We may use the exponential and natural logarithm functions to define the function $g(t) = b^t$ for all $t \in \mathbb{R}$ as follows

$$g(t) = \exp(t \ln b), \quad \text{for all } t \in \mathbb{R}. \tag{14}$$

Use the definition of b^t in (14) to derive formulas for computing

(i) $\dfrac{d}{dt}[b^t]$, and

(ii) $\displaystyle\int b^u\, du$.

Solution:

(i) Applying the Chain Rule, we obtain from (14) that

$$\begin{aligned} g'(t) &= \exp'(t\ln b)\cdot\frac{d}{dt}[t\ln b] \\ &= \exp(t\ln b)\cdot\ln b \\ &= (\ln b)\ b^t, \end{aligned}$$

so that

$$\frac{d}{dt}[b^t] = (\ln b)\ b^t, \qquad \text{for all } t\in\mathbb{R}. \tag{15}$$

(ii) Rewriting (15) we obtain

$$\frac{d}{dt}\left[\frac{1}{\ln b}b^t\right] = b^t, \qquad \text{for all } t\in\mathbb{R},$$

which yields the integration formula

$$\int b^u\, du = \frac{1}{\ln b}b^u + C,$$

where C is an arbitrary constant.

$\square$