Solutions to Assignment #8

Background and Definitions

The exponential function, $\exp\colon \mathbb{R} \to (0,\infty)$, given by $\exp(t) = e^t$, for all $t \in \mathbb{R}$, is the unique solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y; \\ y(0) = 1. \end{cases} \tag{1}$$

1. Use the properties of ln and exp to compute the exact value of $\ln(\sqrt{e})$. Compare your result with the approximation given by a calculator.

   Solution: Compute

   $$\ln(\sqrt{e}) = \ln[e^{1/2}] = \frac{1}{2}\ln e = \frac{1}{2}.$$

   The approximate value given by a calculator is

   $$\ln(\sqrt{e}) \doteq \ln(\sqrt{2.718281828}) \doteq 0.4999999999155633476644925670 2875.$$

   □

2. Let $f(t) = te^{-t^2}$ for all $t \in \mathbb{R}$. Compute $f'(t)$ and $f''(t)$. Determine the intervals on the t–axis for which f is increasing or decreasing, and all local extrema, the values of t for which the graph of f is concave up, and those for which the graph is concave down, and all the inflection points of the graph of f. Sketch the graph of $y = f(t)$.

   Solution: First, we compute $f'(t)$ by applying the product rule and the Chain Rule to get

   $$f'(t) = e^{-t^2} + te^{-t^2}\cdot(-2t), \qquad \text{for all } t \in \mathbb{R},$$

   or

   $$f'(t) = (1-2t^2)e^{-t^2}, \qquad \text{for all } t \in \mathbb{R}, \tag{2}$$

   which we can factor as

   $$f'(t) = 2\left(\frac{1}{\sqrt{2}} + t\right)\left(\frac{1}{\sqrt{2}} - t\right)e^{-t^2}, \qquad \text{for all } t \in \mathbb{R}, \tag{3}$$

   Since $e^{-t^2} > 0$ for all t, it follows from (3) that the sign of $f'(t)$ is determined by the sign of the factors $\dfrac{1}{\sqrt{2}} + t$ and $\dfrac{1}{\sqrt{2}} - t$. Table 1 shows the signs of these factors and the corresponding sign of $f'(t)$. We see from Table 1 that

$\frac{1}{\sqrt{2}}+t$:	$-$	$+$	$+$
$\frac{1}{\sqrt{2}}-t$:	$+$	$+$	$-$
	$-1/\sqrt{2}$	$1/\sqrt{2}$	
$f'(t)$:	$-$	$+$	$-$
$f(t)$:	decreases	increases	decreases

Table 1: Sign of $f'(t)$

$f(t)$ decreases on $(-\infty,-1/\sqrt{2})$ or $(1/\sqrt{2},+\infty)$, and increases on the interval $(-1/\sqrt{2},1/\sqrt{2})$. From this information we also conclude that $f(t)$ is a local minimum when $t=-\sqrt{3}/\sqrt{2}$ and a local maximum when $t=\sqrt{3}/\sqrt{2}$. These extrema are also global extrema and

$$\min f = f(-1/\sqrt{2}) = -\frac{1}{\sqrt{2}}\,e^{-1/2} = -\frac{\sqrt{2}}{2e^{1/2}};$$

similarly, we get that

$$\max f = f(1/\sqrt{2}) = \frac{\sqrt{2}}{2e^{1/2}}.$$

Next, differentiate $f'(t)$ in (2) with respect to t to obtain

$$f''(t) = 4t(t^2-3/2)e^{-t^2}, \qquad \text{for all } t\in\mathbb{R},$$

which factors into

$$f''(t) = 4t\left(t+\frac{\sqrt{3}}{\sqrt{2}}\right)\left(t-\frac{\sqrt{3}}{\sqrt{2}}\right)e^{-t^2}, \qquad \text{for all } t\in\mathbb{R}. \tag{4}$$

Again, since $e^{-t^2}>0$ for all $t\in\mathbb{R}$, in view of (4), the sign of $f''(t)$ is determined by the signs of the factors t, $t+\frac{\sqrt{3}}{\sqrt{2}}$, and $t-\frac{\sqrt{3}}{\sqrt{2}}$. These are shown in Table 2. Table 2 also shows that the graph of $y=f(t)$ is concave down on the intervals $(-\infty,-\sqrt{3}/\sqrt{2})$ or $(0,\sqrt{3}/\sqrt{2})$, and concave up on the intervals $(-\sqrt{3}/\sqrt{2},0)$ or $(\sqrt{3}/\sqrt{2},+\infty)$. We also get that the graph of $y=f(t)$ has inflection points

$t + \dfrac{\sqrt{3}}{\sqrt{2}}$	$-$	$+$	$+$	$+$
t	$-$	$-$	$+$	$+$
$t - \dfrac{\sqrt{3}}{\sqrt{2}}$	$-$	$-$	$-$	$+$
	$-\sqrt{3}/\sqrt{2}$	0	$\sqrt{3}/\sqrt{2}$	
$f''(t)$	$-$	$+$	$-$	$+$
Concavity:	down	up	down	up

Table 2: Concavity of the graph of $y = f(t)$

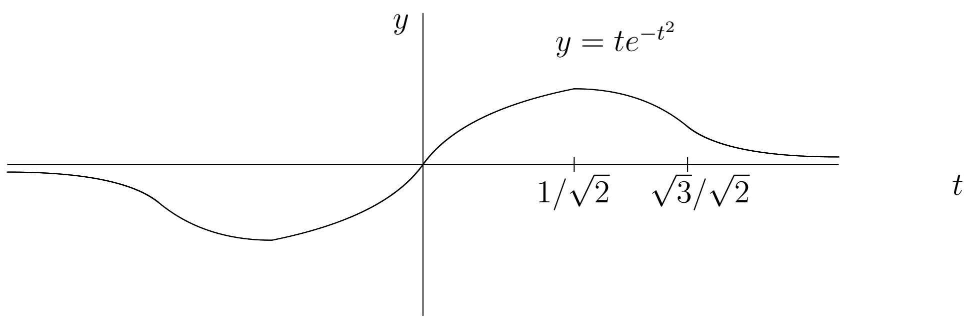

Figure 1: Sketch of graph of $y = te^{-t^2}$

at $-\sqrt{3}/\sqrt{2}$, 0 and $\sqrt{3}/\sqrt{2}$. A sketch of the graph of $y = f(t)$ is shown in Figure 1.

$\square$

3. Let $f(t) = te^{-t^2}$ for all $t \in \mathbb{R}$. For each $b > 0$ compute

$$F(b) = \int_0^b te^{-t^2}\, dt;$$

that is, $F(b)$ is the area under the graph of $y = f(t)$ from $t = 0$ to $t = b$.

Compute $\lim\limits_{b\to\infty} F(b)$. We denote this limit by $\displaystyle\int_0^\infty f(t)\, dt$, and call it the improper integral of f over the interval $(0, \infty)$.

***Solution**:* Make the change of variables $u = -t^2$; so that $du = -2t\ dt$ and

$$\begin{aligned} F(b) &= \int_0^b te^{-t^2}\ dt \\ &= -\frac{1}{2}\int_0^{-b^2} e^u\ du \\ &= \frac{1}{2}\int_{-b^2}^0 e^u\ du \\ &= \frac{1}{2}\left[e^u\right]_{-b^2}^0 \\ &= \frac{1}{2}(1-e^{-b^2}). \end{aligned}$$

Next, compute

$$\begin{aligned} \int_0^\infty f(t)\ dt &= \lim_{b\to\infty} F(b) \\ &= \lim_{b\to\infty} \frac{1}{2}(1-e^{-b^2}) \\ &= \frac{1}{2}. \end{aligned}$$

□

4. Define $f(t) = t^t$, for all $t > 0$, and put $g(t) = \ln[f(t)]$ for all $t > 0$.

   (a) By differentiating g with respect to t, come up with a formula for computing $f'(t)$.

   Note: You will need to apply the Chain Rule when computing $\dfrac{d}{dt}[\ln[f(t)]]$.

   ***Solution**:* Note that, from

   $$g(t) = \ln[f(t)], \tag{5}$$

   we obtain that

   $$g(t) = t\ln t, \tag{6}$$

   for $t > 0$. Differentiating on both sided of (5) we obtain that

   $$g'(t) = \frac{1}{f(t)}\ f'(t),\ \text{ for } t > 0, \tag{7}$$

where we have applied the Chain Rule. On the other hand, applying the product rule on the right–hand side of (6), we obtain

$$g'(t) = \ln t + 1, \text{ for } t > 0. \tag{8}$$

Equating the right–hand sides of (7) and (8) and solving for $f'(t)$ then yields

$$f'(t) = (\ln t + 1)f(t), \text{ for } t > 0. \tag{9}$$

□

(b) Compute $f''(t)$. Does the graph of $y = f(t)$ have any inflection points?

***Solution**:* Differentiate $f'(t)$ in (9), applying the product rule, to obtain

$$f''(t) = \frac{1}{t}f(t) + (\ln t + 1)f'(t), \quad \text{for } t > 0. \tag{10}$$

Next, substitute the expression for $f'(t)$ in (9) into (10) to obtain

$$f''(t) = \left[\frac{1}{t} + (\ln t + 1)^2\right] f(t), \quad \text{for } t > 0. \tag{11}$$

It follows from (11) that $f''(t) > 0$ for $t > 0$, so that the graph of $y = f(t)$ has no inflection points. □

5. Let t_o, r and y_o denote real numbers.

   Verify that $y(t) = y_o e^{r(t-t_o)}$, for $t \in \mathbb{R}$, is the unique solution of the initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = ry; \\ y(t_o) = y_o. \end{cases}$$

***Solution**:* Note that, by the Chain Rule,

$$\begin{aligned} y'(t) &= y_o e^{r(t-t_o)} \cdot \frac{d}{dt}[r(t-t_o)] \\ &= r[y_o e^{r(t-t_o)}] \\ &= ry(t), \end{aligned}$$

for all t. Next, substitute t_o for t to get

$$y(t_o) = y_o e^{r(t-t_o)} = y_o e^{r(t_o-t_o)} = y_o e^0 = y_o.$$

Hence, $y(t) = y_o e^{r(t-t_o)}$ solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ry; \\ y(t_o) = y_o. \end{cases} \tag{12}$$

To show that $y(t) = y_o e^{r(t-t_o)}$, for $t \in \mathbb{R}$, is the only solution to the initial value problem in (12), let $v = v(t)$ be any solution to the initial value problem in (12); then,

$$v'(t) = rv(t), \quad \text{for all } t \in \mathbb{R}, \tag{13}$$

and

$$v(t_o) = y_o. \tag{14}$$

Define the function

$$w(t) = \frac{v(t)}{e^{r(t-t_o)}}, \quad \text{for all } t \in \mathbb{R}. \tag{15}$$

Differentiating the function, w, defined in (15) we obtain, by the quotient rule, that

$$w'(t) = \frac{e^{r(t-t_o)}v'(t) - v(t)re^{r(t-t_o)}}{e^{2r(t-t_o)}}, \quad \text{for all } t \in \mathbb{R}, \tag{16}$$

where we have applied the Chain Rule. Next, use (13) to obtain from (16) that

$$w'(t) = \frac{re^{r(t-t_o)}v(t) - v(t)re^{r(t-t_o)}}{e^{2r(t-t_o)}}, \quad \text{for all } t \in \mathbb{R},$$

so that

$$w'(t) = 0, \quad \text{for all } t \in \mathbb{R}. \tag{17}$$

It follows from (17) that

$$w(t) = c, \quad \text{for all } t \in \mathbb{R}, \tag{18}$$

where c is a constant (see your result in Problem 1 of Assignment 1).

In order to find out the value of c in (18), use (18) and (15) to evaluate

$$c = \frac{v(t_o)}{e^0} = y_o, \tag{19}$$

where we have also used (14).

Next, use (15), (18) and (19) to obtain

$$\frac{v(t)}{e^{r(t-t_o)}} = y_o, \quad \text{for all } t \in \mathbb{R},$$

from which we get

$$v(t) = y_o e^{r(t-t_o)}, \quad \text{for all } t \in \mathbb{R},$$

so that any solution of (12) must in fact be equal to $y(t) = y_o e^{r(t-t_o)}$ for all $t \in \mathbb{R}$. $\square$