Solutions to Assignment #9

1. Assume that a certain strain of *E Coli* bacteria in a culture has a doubling time of about 30 minutes.

   (a) Assuming a Malthusian growth model for the bacteria, give an expression, $N(t)$, for the number of bacteria in the culture at time t, given that at $t = 0$ there are N_o bacteria in the culture.

   ***Solution**:* Assuming a Malthusian growth model given by the differential equation

   $$\frac{dN}{dt} = rN,$$

   where r is the constant *per–capita* growth rate, we have that

   $$N(t) = N_o e^{rt}, \text{ for all } t \geqslant 0, \tag{1}$$

   where

   $$r = \frac{\ln 2}{\tau_2}, \tag{2}$$

   and τ_2 is the doubling time. If t is measured in hours, then

   $$\tau_2 = \frac{1}{2} \text{ hours},$$

   so that, using (2),

   $$r = 2\ln 2 \doteq 1.39, \tag{3}$$

   where the dot on the equal sign in (3) indicates that the left–hand side of (3) is a rational approximation to r. Substituting the approximate value for a in (3) into (1) yields

   $$N(t) \doteq N_o e^{1.39t}, \tag{4}$$

   where t is measured in hours from $t = 0$. □

   (b) How long does it take a thousand bacteria in the culture to produce one million?

   ***Solution**:* Suppose that $N_o = 1000$ in (4). We want to find t so that $N(t) = 10^6$. Thus, using (4) we see that we need to solve the equation

   $$10^3 e^{1.39t} \doteq 10^6, \tag{5}$$

   for t. The equation in (5) is equivalent to

   $$e^{1.39t} \doteq 10^3,$$

which can be solved for t by taking the natural logarithm on both sides of the equation to yield

$$t \doteq \frac{3\ln(10)}{1.38} \doteq 4.97 \text{ hours},$$

or 4 hours and 58 minutes, or nearly 5 hours. □

2. Assume that the bacterial colony described in Problem 1 has an unlimited supply of nutrients conducive to Malthusian growth. Assume also that the bacteria are spherical with a diameter of 10^{-6} meters. Estimate the time that it would take a single bacterium of *E Coli* to grow into a mega–colony to fill the Earth's oceans, seas and bays. Use the estimate given by WolframAlpha® (http://www.wolframalpha.com/) of 1.332×10^{21} liters for the Earth's oceans, seas and bays.

   ***Solution**:* We use the result in (4) with $N_o = 1$ to get that

   $$N(t) \doteq e^{1.39t}, \tag{6}$$

   where t is measured in hours from $t = 0$. Assuming spherical bacteria of radii about 10^{-6} meters, the volume, v_1, of one bacterium is about

   $$v_1 \doteq \frac{4}{3}\pi \left(\frac{1}{2} \times 10^{-6}\right)^3 \text{ cubic meters,}$$

   or

   $$v_1 \doteq 5.24 \times 10^{-19} \text{ cubic meters.}$$

   Using the fact that one cubic meter is equivalent to 1000 liters, we can write the volume of one bacterium in liters as

   $$v_1 \doteq 5.24 \times 10^{-16} \text{ liters.} \tag{7}$$

   Let N denote the number of bacteria needed to reach a volume of 1.332×10^{21} liters, or

   $$Nv_1 = 1.332 \times 10^{21}. \tag{8}$$

   Combining (7) and (8) we see that

   $$N \doteq \frac{1.332 \times 10^{21}}{5.24 \times 10^{-16}} \doteq 2.54 \times 10^{36}. \tag{9}$$

   According to the values predicted by the Malthusian model in (6), the value of N in (9) is achieved when

   $$e^{1.39t} = 2.54 \times 10^{36},$$

or

$$t = \frac{\ln(2.54) + 36\ln(10)}{1.39} \doteq 60.31 \text{ hours},$$

or 2.51 days, or about 2 days, 12 hours and 14 minutes. □

3. Suppose a bacterial colony is growing according to the Malthusian model. Assume that the length of a division cycle corresponds to the doubling time. If the time, t, is measured in units of division cycle divided by $\ln 2$, give a formula for $N(t)$, given that $N(0) = N_o$. By how much does the population increase in one unit of time?

   ***Solution**:* Assume first the $N = N(\tau)$, where τ is measured in an arbitrary continuous time unit. Then, the solution to the Malthus differential equation

$$\frac{dN}{d\tau} = rN,$$

   subject to $N(0) = N_o$, is given by

$$N(t) = N_o e^{r\tau}.$$

   If τ_2 is the doubling time, then $r = \dfrac{\ln(2)}{\tau_2}$ and so

$$N(\tau) = N_o \exp\left(\frac{\ln(2)}{\tau_2}\tau\right) = N_o \exp\left(\frac{\tau}{\tau_2/\ln(2)}\right).$$

   Thus, if t counts the number of division cycles divided by $\ln(2)$, it follows that $t = \dfrac{\tau}{\tau_2/\ln(2)}$; and therefore

$$N(t) = N_o e^t.$$

   Thus, in one division cycle divided by $\ln(2)$, the population increases by

$$\frac{N(1) - N(0)}{N_o} = \frac{N_o e - N_o}{N_o} = e - 1 \approx 1.718$$

   or about 172%. □

4. Assume that the rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time. Let $Q = Q(t)$ denote the amount of the drug in the bloodstream at time t. In Problem 3 of Assignment 1, you applied a conservation principle to derive the differential equation

$$\frac{dQ}{dt} = -kQ, \tag{10}$$

   where k is a positive constant of proportionality, and t is measured in hours.

(a) Solve the differential equation in (10) for the case in which an initial dose of Q_o is injected directly into the blood at time $t = 0$.

Answer: $Q(t) = Q_o e^{-kt}$ for all t. □

(b) Assume that 20% of the initial dose is left in the blood after 3 hours. Write a formula for computing $Q(t)$ for any time t, in hours.

Solution: If $Q(3) = 0.20\ Q_o$, then

$$Q_o e^{-3k} = 0.20\ Q_o \quad \text{or} \quad e^{-3k} = \frac{1}{5}.$$

Thus, $-k = \frac{1}{3} \ln\left(\frac{1}{5}\right)$, or $k = \frac{\ln 5}{3}$. Estimating k to two decimal places we obtain that $k \doteq 0.54$. we then have that

$$Q(t) \doteq Q_o e^{-0.54t}, \tag{11}$$

where t is measured in hours from the initial time $t = 0$. □

(c) What percentage of the initial dosage of the drug is left in the patient's body after 6 hours?

Solution: After 6 hours, the amount of drug present in the patient's blood is

$$Q(6) = Q_o e^{-0.54(6)} \doteq 0.04 Q_o,$$

or 4% of the initial dose. □

5. In a one–compartment dilution experiment, a substance is found dissolved in water in an initial amount Q_o (in moles) in a compartment with constant volume V. Suppose pure distilled water flows into the compartment at a constant rate r (in moles per liter) and that the well–stirred mixture is drained from the tank at the same rate. Suppose that in the experiment the following concentrations of the substance were observed as a function of time:

t[sec]	C[moles/liter]
0	0.024
1	0.011
2	0.0048
3	0.0024
4	0.0010

If $Q_o = 0.1$ mole, find the flow rate r and the volume V.

(*Suggestion:* Plot the natural logarithm of the concentration, $\ln C$, versus time, t, and find the best straight line that fits the data.)

Solution: Applying the conservation principle

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},$$

with

$$\text{Rate of } Q \text{ in} = 0,$$

since distilled water is flowing into the compartment, and

$$\text{Rate of } Q \text{ out} = \frac{Q}{V} r,$$

we obtain that

$$\frac{dQ}{dt} = -\frac{r}{V}\, Q. \tag{12}$$

Solving the differential equation in (12) subject to the initial condition in (12) yields

$$Q(t) = Q_o \exp\left(-\frac{r}{V}\, t\right), \quad \text{for all } t. \tag{13}$$

Dividing the expression in (13) we obtain the following expression for the concentration, $C = C(t)$, of the substance in the solution,

$$C(t) = \frac{Q_o}{V}\, \exp\left(-\frac{r}{V}\, t\right), \quad \text{for all } t. \tag{14}$$

Taking the natural logarithm function on both sided of (14) yields

$$\ln(C) = \ln\left(\frac{Q_o}{V}\right) - \frac{r}{V}\, t, \quad \text{for all } t. \tag{15}$$

Thus, according to (15) plotting $\ln(C)$ versus t should yield a straight line with slope $-\dfrac{r}{V}$ and y–intercept $\ln\left(\dfrac{Q_o}{V}\right)$. Hence, if we want to estimate r and V, in a plot of $\ln(C)$ versus t, we can find the best linear fit to the data and use the fit to get estimates for the slope and y–intercept. Table 1 shows the values of t and $\ln(C)$, the latter rounded up to four decimal places. Figure 1 shows a plot of the data in Table 1 and the least–squares best fitting line obtained using WolframAlpha®. The equation of the best fitting line is

$$y = -3.72804 - 0.78786\, t. \tag{16}$$

t (sec)	$\ln(C)$
0	-3.7297
1	-4.5099
2	-5.3391
3	-6.0323
4	-6.9078

Table 1: Values of t and $\ln(C)$

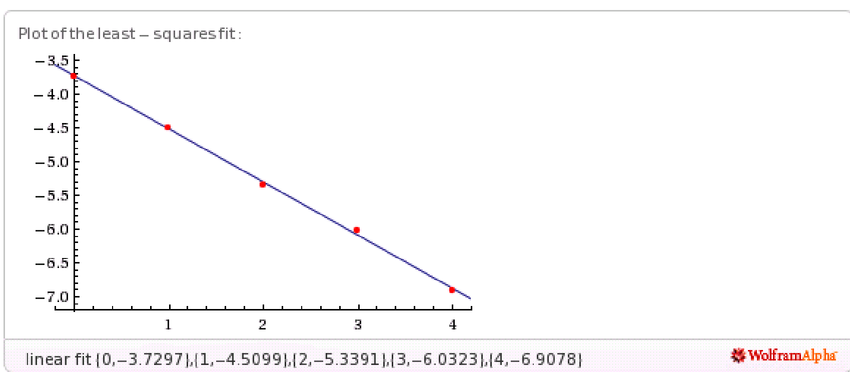

Figure 1: Linear Fit of Data in Table 1

Thus, in view of (16), we see that an estimate for $\ln\left(\dfrac{Q_o}{V}\right)$ is -3.72804 so that

$$\ln\left(\frac{Q_o}{V}\right) \doteq -3.72804,$$

so that

$$\frac{Q_o}{V} \doteq 0.0240. \qquad (17)$$

Solving for V in (17) and using the value of 0.1 mole for Q_o, we obtain from (17) that

$$V \doteq 4.17 \ \text{ liters}. \qquad (18)$$

To find r, use the equation of the best–fitting line in (16) to obtain the estimate

$$\frac{r}{V} \doteq 0.78786. \qquad (19)$$

Thus, using the estimate for V in (18), we obtain from (19) that $r \doteq 3.29$ sec^{-1}.
□