Solutions to Review Problems for Exam #1

1. Water leaks out a barrel at a rate proportional to the square root of the depth of the water in the barrel at that time. Derive a differential equation for the depth, $h(t)$, of water in the barrel at time t, given that the cross–sectional area of the barrel is a constant A.

   ***Solution**:* Let $Q(t)$ the volume of water in the barrel. Then,

   $$Q(t) = Ah(t), \qquad \text{for all } t.$$

   The conservation principle in this case states that

   $$\frac{dQ}{dt} = \text{Rate of } Q \text{ in } - \text{Rate of } Q \text{ out},$$

   where

   $$\text{Rate of } Q \text{ in } = 0$$

   and

   $$\text{Rate of } Q \text{ out } = k\sqrt{h},$$

   for some positive constant of proportionality k. We then have that

   $$\frac{dQ}{dt} = 0 - k\sqrt{h},$$

   where

   $$\frac{dQ}{dt} = \frac{d}{dt}(Ah) = A\frac{dh}{dt},$$

   since A is constant.

   Consequently,

   $$A\frac{dh}{dt} = -k\sqrt{h},$$

   or

   $$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h}.$$

   □

2. A compartment has a fixed volume, V, of isopropyl alcohol solution. A 75% solution of isopropyl alcohol is introduced into the compartment at a rate of $F = 0.1$ liters per minute. Assume that the a well–stirred mixture of the solution flows out of the compartment at the same rate, F.

(a) Derive a differential equation for the concentration of alcohol, in percent volume, at any time t.

Solution: Let $Q(t)$ denote the volume of isopropyl alcohol in the solution at time t. The conservation principle implies that

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in } - \text{Rate of } Q \text{ out},$$

where

$$\text{Rate of } Q \text{ in } = c_o F,$$

where $c_o = 0.75$ is the concentration of isopropyl alcohol flowing into the solution at a rate F, and

$$\text{Rate of } Q \text{ out } = c(t)F,$$

where

$$c(t) = \frac{Q(t)}{V} \tag{1}$$

is the concentration of isopropyl alcohol in the solution at time t. Hence,

$$\frac{dQ}{dt} = Fc_o - Fc, \tag{2}$$

Now, solving for Q in (1) we obtain that

$$Q = Vc,$$

so that, since V is constant,

$$\frac{dQ}{dt} = V\frac{dc}{dt},$$

and substituting into the equation in (2),

$$V\frac{dc}{dt} = Fc_o - Fc; \tag{3}$$

so that, dividing the equation in (3) by V,

$$\frac{dc}{dt} = \frac{F}{V}(c_o - c). \tag{4}$$

□

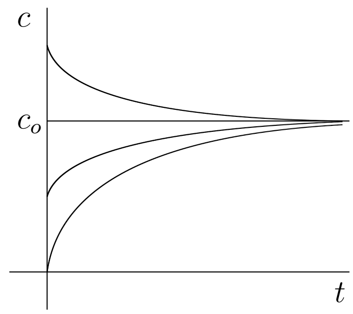

Figure 1: Sketch of possible solutions of (4)

(b) Sketch possible solutions of the equation.

***Solution**:* From equation (4) we see that $c'(t) > 0$ for $0 < c < c_o$; thus, $c(t)$ increases for $0 < c < c_o$. Similarly, since $c'(t) < 0$ for $c > c_o$, $c(t)$ decreases for $c > c_o$. Observe also that, if $c = c_o$, then $c'(t) = 0$ for all t; so that, c remains constant at $c = c_o$.

Next, we determine the concavity of the graph of c as a function of t.

Taking the derivative on both sides of (4) we obtain

$$\begin{aligned} c''(t) &= \frac{d}{dt}\left(\frac{F}{V}(c_o - c)\right) \\ &= -\frac{F}{V}\frac{dc}{dt}, \end{aligned}$$

where we have used the Chain Rule. Thus, using (4),

$$\begin{aligned} c''(t) &= -\frac{F}{V}\cdot\frac{F}{V}(c_o - c) \\ &= -\frac{F^2}{V^2}(c_o - c). \end{aligned}$$

Hence, $c''(t) < 0$ for $0 < c < c_o$; so that, the graph of c as a function of t is concave down for $0 < c < c_o$. Similarly, since $c''(t) > 0$ for $c > c_o$, the graph of c as a function of t is concave up for $c > c_o$.

The graphs of possible solutions of (3) are shown in Figure 1. □

3. A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Salt water enters the tank at a rate of 9 gal/hr with a salt

concentration of 3 lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, derive a differential equation satisfied by the amount of salt in the solution in the tank.

***Solution**:* Let V_o denote the initial volume of solution in the tank; that is, $V_o = 600$ gallons. Let F_1 denote the flow rate of water into the tank; so that, $F_1 = 9$ gal/hr. Let F_2 denote the flow rate of solution out of the tank; so that, $F_2 = 6$ gal/hr. Thus, the net rate of flow of solution into the tank is $F_1 - F_2 = 3$ gal/hr.

Let $Q(t)$ denote the amount of salt in the solution at time t. Then, the conservation principle implies that

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in } - \text{Rate of } Q \text{ out}, \tag{5}$$

where

$$\text{Rate of } Q \text{ in } = c_1 F_1, \tag{6}$$

$c_1 = 3$ lbs per gallon is the concentration of salt in the solution flowing into the tank, and

$$\text{Rate of } Q \text{ out } = c(t) F_2, \tag{7}$$

where

$$c(t) = \frac{Q(t)}{V_o + (F_1 - F_2)t}. \tag{8}$$

since the volume of solution in the tank is increasing at a constant rate of $F_1 - F_2 = 3$ gal/hr.

Substituting (8) into (7) then yields

$$\text{Rate of } Q \text{ out } = F_2 \cdot fracQ(t)V_o + (F_1 - F_2)t. \tag{9}$$

Combining (9), (6) and (5), we obtain

$$\frac{dQ}{dt} = c_1 F_1 - F_2 \cdot \frac{Q(t)}{V_o + (F_1 - F_2)t},$$

or

$$\frac{dQ}{dt} = 27 - 6 \cdot \frac{Q(t)}{600 + 3t},$$

in units of pounds per hour. □

4. Suppose that $y = y(t)$ is a solution of the differential equation

$$\frac{dy}{dt} = -2ty \tag{10}$$

(a) Assume that $y(t) > 0$ for all t. Determine the values of t for which $y(t)$ increases or decreases.

***Solution**:* From the differential equation in (10) we obtain that

$$y'(t) = -2ty(t), \qquad \text{for all } t; \tag{11}$$

so that, since $y(t) > 0$ for all t, $y'(t) > 0$ for $t < 0$ and $y'(t) < 0$ for $t > 0$. Thus, $y(t)$ increases for $t < 0$ and it decreases for $t > 0$. $\square$

(b) Compute y'' in terms of t and y, and determine the values of t for which the graph of $y = y(t)$ is concave up or concave down.

***Solution**:* Differentiate on both sides of the equation in (11) with respect to t, using the product rule, to get

$$\begin{aligned} y''(t) &= \frac{d}{dt}[-2ty(t)] \\ &= -2y(t) - 2ty'(t); \end{aligned}$$

so that, substituting the expression for $y'(t)$ in (11),

$$\begin{aligned} y''(t) &= -2y(t) - 2t(-2ty(t)) \\ &= -2y(t) + 4t^2y(t) \\ &= (4t^2 - 2)y(t), \end{aligned}$$

which we can factor further,

$$y''(t) = 4\left(t^2 - \frac{1}{2}\right)y(t),$$

or

$$y''(t) = 4\left(t + \frac{1}{\sqrt{2}}\right)\left(t - \frac{1}{\sqrt{2}}\right)y(t). \tag{12}$$

Since we are assuming that $y(t) > 0$ for all t, it follows from (12) that the sign of $y''(t)$ is determined by the product of the signs of the expressions $\left(t + \frac{1}{\sqrt{2}}\right)$ and $\left(t - \frac{1}{\sqrt{2}}\right)$. These are displayed in Table 1. The table also displays the intervals in which the graph of y is concave up and that in which the graph is concave down. $\square$

$t+\dfrac{1}{\sqrt{2}}$	$-$		$+$		$-$
$t-\dfrac{1}{\sqrt{2}}$	$-$		$-$		$+$
		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
$y''(t)$	$+$		$-$		$+$
graph of $y(t)$	concave–up		concave–down		concave–up

Table 1: Concavity of the graph of $y(t)$

(c) Given that $y(0) = 1$, use the qualitative information obtained in the previous parts to sketch the graph of $y = y(t)$.

***Solution**:* Figure 2 shows a sketch of the the graph of y. Observe that the graph has a local maximum at $t = 0$ and inflection points at $t = -\dfrac{1}{\sqrt{2}}$ and $t = \dfrac{1}{\sqrt{2}}$. □

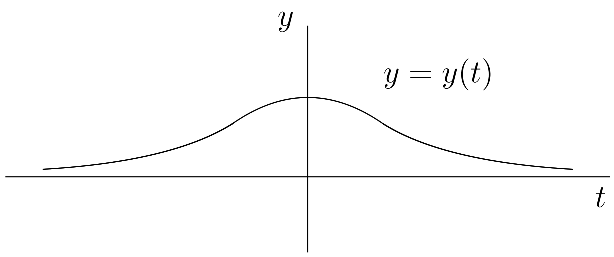

Figure 2: Sketch of graph of $y = y(t)$

5. Sketch possible solutions of the differential equation

$$\frac{dy}{dt} = (y-1)(y-2). \tag{13}$$

***Solution**:* Set $g(y) = k(40 - y)(80 - y)$, or

$$g(y) = (y-1)(y-2), \tag{14}$$

so that the differential equation in (13) becomes

$$\frac{dy}{dt} = g(y). \tag{15}$$

We have sketched that graph of g as a function of y in Figure 3. Looking at

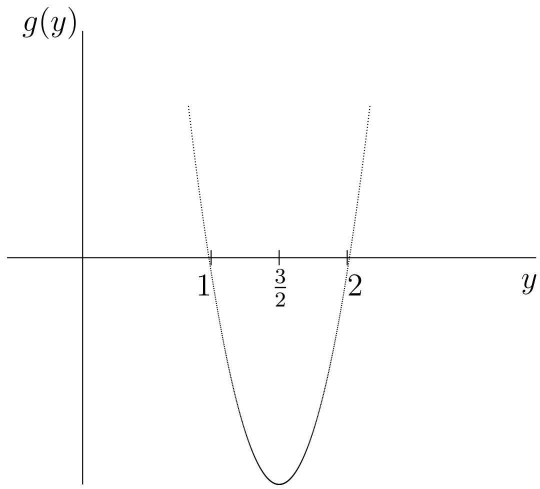

Figure 3: Sketch of graph of g as a function of y

the graph of g as a function of y in Figure 3 we see that $g(y) > 0$ for $y < 1$, or $y > 2$. Consequently, in view of (15), $y(t)$ increases for $y < 1$ and $y > 2$. Similarly, we see that $y(t)$ decreases for $1 < y < 2$. When $y = 1$ or $y = 2$, $y'(t) = 0$ for all t. Consequently, $y(t) = 1$ for all t, and $y(t) = 2$ for all t, are possible solutions of the differential equation. These are pictured in Figure 4.

To obtain sketched of other possible solutions, we compute y'' by differentiating on both sides of (15) to get

$$y'' = g'(y) \cdot \frac{dy}{dt},$$

where we have used the Chain Rule. Consequently, using (15) again, we get that

$$y'' = g'(y) \cdot g(y).$$

$g'(y)$	$-$	$-$	$+$	$+$
$g(y)$	$+$	$-$	$-$	$+$
	1	3/2	2	
$y''(t)$	$-$	$+$	$-$	$+$
graph of $y(t)$	concave–down	concave–up	concave–down	concave–up

Table 2: Concavity of the graph of $y(t)$

Thus, the sign of $y''(t)$ is determined by the product of the signs of $g(y)$ and $g'(y)$. These are shown in Table 2. The signs for $g'(y)$ in Table 2 were obtained by observing that $g(y)$ degrease for $y < 1.5$; so that, $g'(y) < 0$ for $y < 1.5$; and $g(y)$ increases for $y > 1.5$; so that, $g'(y) > 0$ for $y > 1.5$. The concavity of the graph for $y(t)$ is also shown in Table 2.

Putting together the qualitative information that we have obtained thus far, we get the sketches of possible solutions of the equation in (15) shown in Figure 4. □

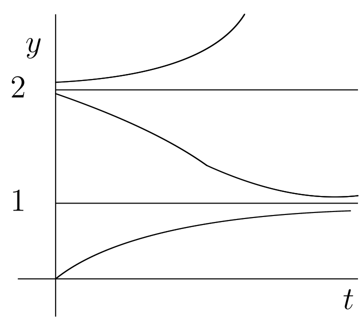

Figure 4: Possible Solutions to the equation in (13)

6. In a chemical reaction

$$A + B \to C,$$

let $y(t)$ denote the concentration of the product C at time t. Assume that y is a differentiable function of t. If C_A denote the initial concentraction of reactant A

and C_B the initial concentration of reactant B, the Law of Mass Action states that

$$\frac{dy}{dt} = k(C_A - y)(C_B - y), \tag{16}$$

where k is a positive constant of proportionality.

Sketch possible solutions of (16) for the case in which $C_A = 40$ and $C_B = 80$.

***Solution**:* Rewrite the equation in (16) with the given values

$$\frac{dy}{dt} = k(40 - y)(80 - y), \tag{17}$$

Set $g(y) = k(40 - y)(80 - y)$, or

$$g(y) = k(y - 40)(y - 80), \tag{18}$$

so that the differential equation in (17) becomes

$$\frac{dy}{dt} = g(y). \tag{19}$$

We have sketched that graph of g as a function of y in Figure 5. Looking at

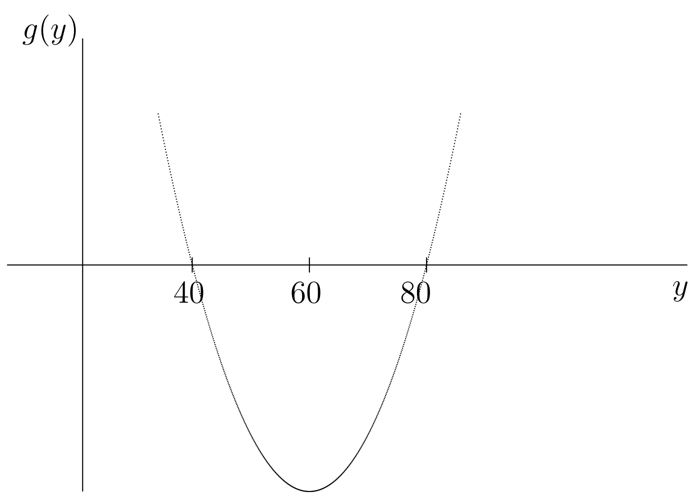

Figure 5: Sketch of graph of g as a function of y

	$y<40$	$40<y<60$	$60<y<80$	$y>80$
$g'(y)$	$-$	$-$	$+$	$+$
$g(y)$	$+$	$-$	$-$	$+$
	40	60	80	
$y''(t)$	$-$	$+$	$-$	$+$
graph of $y(t)$	concave–down	concave–up	concave–down	concave–up

Table 3: Concavity of the graph of $y(t)$

the graph of g as a function of y in Figure 5 we see that $g(y) > 0$ for $y < 40$, or $y > 80$. Consequently, in view of (19), $y(t)$ increases for $y < 40$ and $y > 80$. Similarly, we see that $y(y)$ decreases for $40 < y < 80$. When $y = 40$ or $y = 80$, $y'(t) = 0$ for all t. Consequently, $y(t) = 40$ for all t, and $y(t) = 80$ for all t, are possible solutions of the differential equation. These are pictured in Figure 6.

To obtain sketched of other possible solutions, we compute y'' by differentiating on both sides of (19) to get

$$y'' = g'(y) \cdot \frac{dy}{dt},$$

where we have used the Chain Rule. Consequently, using (19) again, we get that

$$y'' = g'(y) \cdot g(y).$$

Thus, the sign of $y''(t)$ is determined by the product of the signs of $g(y)$ and $g'(y)$. These are shown in Table 3. The signs for $g'(y)$ in Table 3 were obtained by observing that $g(y)$ degrease for $y < 60$; so that, $g'(y) < 0$ for $y < 60$; and $g(y)$ increases for $y > 60$; so that, $g'(y) > 0$ for $y > 60$. The concavity of the graph for $y(t)$ is also shown in Table 3.

Putting together the qualitative information that we have obtained thus far, we get the sketches of possible solutions of the equation in (19) shown in Figure 6.

□

7. The following equation models the growth of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75. \tag{20}$$

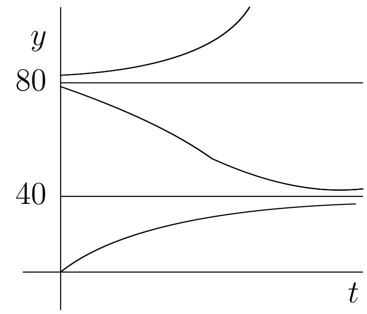

Figure 6: Possible Solutions to the equation in (17)

Sketch possible solutions of the differential equation.

Solution: Set $g(N) = 2N - 0.01N^2 - 75$, or

$$g(N) = -\frac{1}{100}[N^2 - 200N + 7500],$$

which can be factored as

$$g(N) = -\frac{1}{100}(N-50)(N-150). \tag{21}$$

The differential equation in (20) can then be rewritten as

$$\frac{dN}{dt} = g(N). \tag{22}$$

Note that, according to (21), $g(50) = 0$ and $g(150) = 0$; thus, $N'(t) = 0$ for all t is the initial condition N_o is wither 50 or 150. Hence, the constant functions $N(t) = 50$ for all t, and $N(t) = 150$, for all t, are possible solutions of the equation in (20). These are sketched in Figure 8.

First, we determine where a solution $N(t)$ of (22) increases or decreases. In order to do so, we use the information given by the equation in (22) together with the sketch of the graph of g versus N given in Figure 7. Inspection of the sketch in Figure 7 shows that $g(N)$ is positive for $50 < N < 150$; consequently, according to (22) $N'(t) > 0$ for $50 < N < 150$; thus, $N(t)$ increases for $50 < N < 150$. Similarly, we see from the graph in (7) that $N'(t) < 0$ for $0 < N < 50$ or $N > 150$; hence, $N(t)$ decreases for $0 < N < 50$ or $N > 150$.

Next, we determine the concavity of the solutions of (22) by looking at $N''(t)$. To do so, we differentiate the equation in (22) with respect to t on both sides

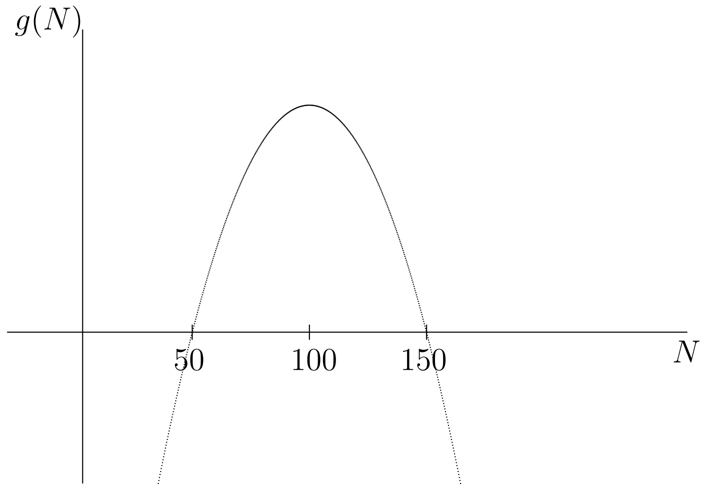

Figure 7: Sketch of graph of g as a function of N

to get

$$\begin{aligned} N''(t) &= \frac{d}{dt}[g(N)] \\ &= g'(N) \cdot \frac{dN}{dt}, \end{aligned}$$

where we have used the Chain Rule. Consequently, using (22),

$$N''(t) = g'(N) \cdot g(N). \tag{23}$$

It follows from (23) that the sign of $N''(t)$ is determined by the product of the signs of $g(N)$ and $g'(N)$. Table 4 shows the signs of $g(N)$, $g'(N)$, $N''(t)$ and the concavity of the graph of N. Figure 8 shows sketches of possible solutions to the equation subject to the initial conditions $N(0) = N_o$, where N_o is 40, 50, 60, 150, and 160. □

8. Use the Chain Rule to show that $y(t) = y_o \exp(F(t))$, where F is the antiderivative of f with $F(0) = 0$, is a solution of the initial value problem: $\dfrac{dy}{dt} = f(t)y$, $y(0) = y_o$.

$g'(N)$	$+$	$+$	$-$	$-$
$g(N)$	$-$	$+$	$+$	$-$
	50	100	150	
$N''(t)$	$-$	$+$	$-$	$+$
graph of $N(t)$	concave–down	concave–up	concave–down	concave–up

Table 4: Concavity of the graph of $N(t)$

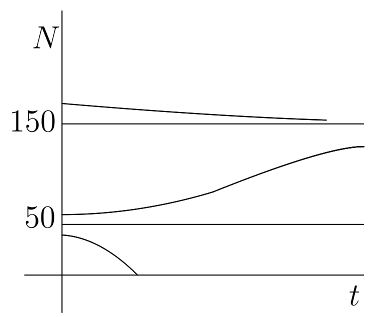

Figure 8: Possible solutions of the equation (20)

Solution: Apply the Chain Rule to obtain

$$\begin{aligned}\frac{dy}{dt} &= y_0 \exp'(F(t))F'(t)\\ &= y_0 \exp(F(t))f(t)\\ &= f(t)[y_0 \exp(F(t))],\end{aligned}$$

which shows that

$$\frac{dy}{dt} = f(t)y.$$

Next, compute

$$y(0) = y_o \exp(F(0)) = y_o \exp(0) = y_o.$$

Hence, if $F\colon I \to \mathbb{R}$ is differentiable over some open interval I which contains 0, with $F' = f$ on I , and $F(0) = 0$, then $y(t) = y_o \exp(F(t))$ for $t \in I$ solves the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t)y \\ y(0) = y_o. \end{cases}$$

□

9. Evaluate the following integrals

$$\text{(a)} \quad \int_0^1 \frac{e^{-x}}{2-e^{-x}} \mathrm{d}x \qquad \text{(b)} \quad \int \frac{1}{x\ln x} \mathrm{d}x$$

$$\text{(c)} \quad \int_1^2 \frac{\ln x}{x} \mathrm{d}x \qquad \text{(d)} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \mathrm{d}x$$

***Solution**:*

(a) Make the change of variables $u = 2 - e^{-x}$, so that $du = e^{-x}\ dx$. Then,

$$\int_0^1 \frac{e^{-x}}{2-e^{-x}}\ dx \quad = \quad \int_1^{2-e^{-1}} \frac{1}{u}\ du = \ln(2 - e^{-1}),$$

by the definition of the natural logarithm function.

(b) Make the change of variables $u = \ln x$, so that $du = \dfrac{1}{x}\ dx$ and

$$\begin{aligned} \int \frac{1}{x\ln x}\ dx &= \int \frac{1}{u}\ du \\ &= \ln|u| + C \\ &= \ln|\ln x| + C. \end{aligned}$$

(c) Make the change of variables $u = \ln x$, so that $du = \dfrac{1}{x}\ dx$ and

$$\int_1^2 \frac{\ln x}{x}\ dx \quad = \quad \int_0^{\ln 2} u\ du = \frac{1}{2}[\ln 2]^2.$$

(d) Make the change of variables $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$, and

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int e^u\,du \\ &= 2e^u + C \\ &= 2e^{\sqrt{x}} + C.\end{aligned}$$

$\square$

10. Solve the initial value problem

$$\begin{cases}\dfrac{dy}{dt} = \dfrac{1}{t+\sqrt{t}}; \\ \\ y(1) = 2\ln(2),\end{cases}$$

for $t > 0$.

***Solution**:* Compute

$$y(t) = 2\ln(2) + \int_1^t \frac{1}{\tau+\sqrt{\tau}}\,d\tau, \qquad \text{for } t > 0. \tag{24}$$

We evaluate the integral in (24) by making the change of variables $u = \sqrt{\tau}$. Then, $du = \dfrac{1}{2\sqrt{\tau}}\,d\tau$, so that $d\tau = 2u\,du$. Substituting into the integral in (24) we obtain that

$$\begin{aligned}\int_1^t \frac{1}{\tau+\sqrt{\tau}}\,d\tau &= \int_1^{\sqrt{t}} \frac{2u}{u^2+u}\,du \\ &= 2\int_1^{\sqrt{t}} \frac{1}{u+1}\,du.\end{aligned} \tag{25}$$

Making a further change of variables, $v = u + 1$, in the last integral in (25) we obtain

$$\begin{aligned}\int_1^t \frac{1}{\tau+\sqrt{\tau}}\,d\tau &= 2\int_2^{1+\sqrt{t}} \frac{1}{v}\,dv \\ &= 2\ln(1+\sqrt{t}) - 2\ln(2).\end{aligned} \tag{26}$$

Substituting the result in (26) into (24) we obtain

$$y(t) = 2\ln(2) + 2\ln(1+\sqrt{t}) - 2\ln(2), \qquad \text{for } t > 0,$$

from which we get that

$$y(t) = 2\ln(1+\sqrt{t}), \qquad \text{for } t > 0.$$

$\square$