Solutions to Review Problems for Final Exam

1. An initial population of 50,000 inhabits a microcosm with carrying capacity of 100,000. Suppose that, after five years, the population increases to 60,000. Determine the intrinsic growth rate of the population.

   Solution: We assume that the growth of the population is governed by the Logistic equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$
   where $K = 10^5$ in this case.

   The solution of the differential equation in (1) subject to the initial condition $N(0) = N_o$ is given by
$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}}, \quad \text{for } t \geqslant 0. \tag{2}$$
   For the situation at hand, $N_0 = 5 \times 10^4$.

   Given that $N(5) = 6 \times 10^4$, we would like to determine the value of the intrinsic growth rate, r. In order to do this, we solve (2) for r. Writing N for $N(t)$ in (2) and taking reciprocals on both sides of the equation we obtain
$$\frac{1}{N} = \frac{N_o + (K - N_o)e^{-rt}}{N_o K},$$
   which can be re–written as
$$\frac{1}{N} = \frac{1}{K} + \left(\frac{1}{N_o} - \frac{1}{K}\right)e^{-rt}. \tag{3}$$
   The equation in (3) can now be solved for e^{-rt} to yield
$$e^{-rt} = \frac{\dfrac{1}{N} - \dfrac{1}{K}}{\dfrac{1}{N_o} - \dfrac{1}{K}},$$
   or
$$e^{-rt} = \frac{N_o}{N} \cdot \frac{K - N}{K - N_o}. \tag{4}$$
   Taking reciprocals on both sides of (4) yields
$$e^{rt} = \frac{N(K - N_o)}{N_o(K - N)}. \tag{5}$$

Taking the natural logarithm on both sides of (5) and solving for r, we obtain

$$r = \frac{1}{t} \ln \left(\frac{N(K - N_o)}{N_o(K - N)} \right). \tag{6}$$

Substituting the values $N = 6 \times 10^4$, $N_o = 5 \times 10^4$, $K = 10^5$, and $t = 5$ into (6) yields

$$r = \frac{1}{5} \ln \left(\frac{3}{2} \right).$$

□

2. Hydrocoden bitartrate is prescription drug used as a cough suppressant and pain reliever. Assume the drug is eliminated from the body by a natural decay process with half–life of 3.8 hours. The usual dose is 10 mg every 6 hours.

   (a) Use a conservation principle to derive a differential equation satisfied by the amount $Q(t)$ of the drug in the patient after a dose.

   ***Solution**:* Model the patient's bloodstream as a compartment of fixed volume. Let $Q = Q(t)$ denote the amount of drug in the compartment at time t. Apply the conservation principle

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}, \tag{7}$$

   where

$$\text{Rate of } Q \text{ in } = 0, \tag{8}$$

$$\text{Rate of } Q \text{ out } = \lambda Q, \tag{9}$$

   with $\lambda > 0$ being a constant of proportionality.

   Combining the equations (7)–(9), we obtain the differential equation

$$\frac{dQ}{dt} = -\lambda Q. \tag{10}$$

   □

   (b) Assume that the amount of the drug in the patient prior to the dose is Q_o and that the drug is absorbed immediately. Give a formula for computing $Q(t)$, where t measures the length of time after the dose.

   ***Solution**:* The solution to the differential equation in (10) subject to the initial condition $Q(0) = Q_o$ is

$$Q(t) = Q_o e^{-\lambda t}, \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

The rate constant, λ, is related to the half–life, τ_2, by means of the equation

$$\lambda = \frac{\ln 2}{\tau_2}, \tag{12}$$

where $\tau_2 = 3.8$ hours in this case. Combining (11) and (12) yields the formula

$$Q(t) = Q_o e^{-\frac{\ln 2}{\tau_2}t}, \quad \text{for all } t \in \mathbb{R},$$

or

$$Q(t) = \frac{Q_o}{2^{t/3.8}},$$

where t is measured in hours. □

3. Suppose that alcohol is introduced into a 2-liter beaker, which initially contains distilled water, at a rate of 0.1 liners per minute. Assume that the a well–mixed mixture is removed from the beaker at the same rate.

   (a) Derive a differential equation for the concentration of alcohol in percent volume at any time t.

   ***Solution**:* The beaker is a compartment of fixed volume $V = 2$ liters. Let $Q = Q(t)$ denote the volume of alcohol in the compartment at time t. Apply the conservation principle

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}. \tag{13}$$

   Let F denote the rate at which alcohol flows into the beaker; in this case, $F = 0.1$ liters per minute. Then,

$$\text{Rate of } Q \text{ in } = F. \tag{14}$$

   Setting

$$c(t) = \frac{Q(t)}{V}, \quad \text{for all } t, \tag{15}$$

   the concentration of alcohol in the beaker at time t, in percent volume, we have that

$$\text{Rate of } Q \text{ out } = c(t)F,$$

   or

$$\text{Rate of } Q \text{ out } = \frac{F}{V}Q, \tag{16}$$

by virtue of (15).

Combining the equations (13), (14) and (16), we obtain the differential equation

$$\frac{dQ}{dt} = F - \frac{F}{V}\,Q, \tag{17}$$

for the volume of alcohol in the beaker at time t.

Next, divide the equation in (17) by V, and use (15) to obtain the differential equation

$$\frac{dc}{dt} = \frac{F}{V} - \frac{F}{V}\,c,$$

or

$$\frac{dc}{dt} = -\frac{F}{V}\,(c-1)\,. \tag{18}$$

□

(b) How long will it take for the concentration of alcohol to reach 50%?

Solution: In order to answer this question, we solve the differential equation in (18) subject to the initial condition $c(0) = 0$, since the beaker starts one with 2 liters of distilled water. The solution to this initial value problem is

$$c(t) = 1 - e^{-\frac{F}{V}\,t}, \tag{19}$$

where t is measured in minutes.

Next, we solve the equation

$$c(t) = 0.5,$$

where $c(t)$ is given by (19), to obtain

$$t = \frac{V}{F}\,\ln 2 \doteq 13.86 \text{ minutes.}$$

□

4. A patient who has asthma is given a continuous infusion of theophylline to relax and open the air passages in the patient's lungs. The desired steady state level of theophylline in the patient's bloodstream is 15 milligrams per liter. The average half–life of theophylline is about four hours. Assume the bloodstream's volume in the patient is about 5.6 liters.

(a) Determine the necessary infusion rate needed to maintain the theophylline level around 15 mg/L.

***Solution**:* Let r_i denote the infusion rate and $Q(t)$ denote the amount of theophylline in the patient's bloodstream at time t. Applying the conservation principle to Q we have that

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}, \tag{20}$$

where

$$\text{Rate of } Q \text{ in } = r_i, \tag{21}$$

and

$$\text{Rate of } Q \text{ out } = \lambda Q, \tag{22}$$

with $\lambda > 0$ being a constant of proportionality.

Substituting (21) and (22) into (20) then yields the first–order differential equation

$$\frac{dQ}{dt} = r_i - \lambda Q. \tag{23}$$

To determine λ we use the half-life, $\tau_{1/2}$, of the drug in the patient's bloodstream and the formula

$$\lambda = \frac{\ln 2}{\tau_{1/2}}. \tag{24}$$

In this case we take $\tau_{1/2} = 4$ hours.

Diving the equation in (23) by the volume $V = 5.6$ L of the patient's bloodstream, we obtain the differential equation

$$\frac{dc}{dt} = \frac{r_i}{V} - \lambda c, \tag{25}$$

for the concentration,

$$c = \frac{Q}{V}, \tag{26}$$

of theophylline in the patient's bloodstream.

Rewriting the equation in (25) as

$$\frac{dc}{dt} = -\lambda \left(c - \frac{r_i}{\lambda V} \right), \tag{27}$$

we see that

$$\overline{c} = \frac{r_i}{\lambda V} \tag{28}$$

is an equilibrium point of the equation in (27). Note also that, by the Principle of Linearized Stability, $\overline{c}$ is asymptotically stable. This corresponds to the desired steady state level of theophylline in the body, which is given in the problem as 15 mg/L. We therefore set $\overline{c}$ in (28) equal to 15 mg/L to get

$$\frac{r_i}{\lambda V} = 15 \text{ mg/L},$$

from which we get the expression

$$r_i = \lambda V \left(15 \ \frac{\text{mg}}{\text{L}}\right), \tag{29}$$

for the infusion rate.

Substituting the expression for λ in (24) in (29) then yields

$$r_i = \frac{\ln 2}{\tau_{1/2}} \left(15 \ \frac{\text{mg}}{\text{L}}\right) V. \tag{30}$$

Finally, using the value 5.6 L for V and 4 hr for $\tau_{1/2}$ in (30), we obtain the infusion rate

$$r_i = \frac{\ln 2}{4 \text{ hr}} \left(15 \ \frac{\text{mg}}{\text{L}}\right) (5.6 \text{ L}) = 14.56 \ \frac{\text{mg}}{\text{hr}}.$$

□

(b) Determine how long it will take for the concentration of theophylline in the patient's body to reach 10 mg/L, assuming that there is no theophylline in the patient's body at time $t = 0$.

***Solution**:* We solve the initial value problem

$$\begin{cases} \dfrac{dc}{dt} = -\lambda\,(c - \overline{c})\,; \\ c(0) = 0, \end{cases} \tag{31}$$

where $\overline{c}$ is given by (28), and which we take to be 15 mg/L, according to the result of the previous part.

The general solution of the differential equation in (31) is

$$c(t) = \overline{c} + \gamma e^{-\lambda t}, \qquad \text{for all } t, \tag{32}$$

where γ is an arbitrary constant.

To find γ, use the initial condition in (31) to get from (32) that

$$\overline{c} + \gamma = 0,$$

from which we get that
$$\gamma = -\overline{c}. \tag{33}$$
Substituting the value of γ in (33) into (32) yields
$$c(t) = \overline{c}(1 - e^{-\lambda t}), \quad \text{for all } t, \tag{34}$$
where λ is given in (24).

To answer the question posed in this part of the problem, we find t such that $c(t) = 10$ mg/L, where $c(t)$ is given in (34) with $\overline{c} = 15$ mg/L; thus, we need to solve the equation
$$15(1 - e^{-\lambda t}) = 10,$$
or
$$1 - e^{-\lambda t} = \frac{2}{3},$$
or
$$e^{-\lambda t} = \frac{1}{3}. \tag{35}$$
Taking the natural logarithm function on both sides of (35) and solving for t yields
$$t = \frac{\ln 3}{\lambda}, \tag{36}$$
where λ is given in (24). We then obtain from (36) that
$$t = \frac{\ln 3}{\ln 2}\tau_{1/2} = \frac{\ln 3}{\ln 2}(4 \text{ hr}) \approx 6.34 \text{ hr}.$$

□

5. Estimate $\displaystyle\int_0^{0.047} e^{-t^2}\, dt$ and determine the accuracy of your estimate. Explain the reasoning leading to your answer.

***Solution**:* Let $f(x) = \displaystyle\int_0^{x} e^{-t^2}\, dt$, for all $x \in \mathbb{R}$. We want to estimate $f(0.047)$. We will use the linear approximation to f at $a = 0$:
$$L(x; a) = f(a) + f'(a)(x - a), \quad \text{for } x \in \mathbb{R},$$
or
$$L(x; 0) = f(0) + f'(0)x, \quad \text{for } x \in \mathbb{R}, \tag{37}$$

where $f(0) = \int_0^0 e^{-t^2}\, dt = 0$, and, by the Fundamental Theorem of Calculus,

$$f'(x) = e^{-x^2}, \quad \text{for } x \in \mathbb{R}; \tag{38}$$

so that, $f'(0) = 1$. We then obtain from (37) that

$$L(x; 0) = x, \quad \text{for all } x.$$

We can therefore make the approximation

$$f(x) \approx x, \quad \text{for } x \text{ very close to } 0,$$

or

$$\int_0^x e^{-t^2}\, dt \approx x, \quad \text{for } x \text{ very close to } 0.$$

In particular,

$$\int_0^{0.047} e^{-t^2}\, dt \approx 0.047. \tag{39}$$

To determine the accuracy of the estimate in (39), we estimate the error in the linear approximation, $E_1(x; a)$, using

$$|E_1(x; a)| \leqslant \frac{M}{2}|x - a|^2, \tag{40}$$

where $a = 0$, and M is a positive value for which

$$|f''(x)| \leqslant M, \quad for x \in I,$$

where I is an interval that contains a.

Differentiate the expression for f' in (38) to get

$$f''(x) = -2xe^{-x^2}, \quad \text{for } x \in \mathbb{R};$$

so that

$$|f''(x)| = 2|x|e^{-x^2}, \quad \text{for all } x,$$

from which we get that

$$|f''(x)| \leqslant 2|x|, \quad \text{for all } x,$$

because $e^{-x^2} \leqslant 1$ for all x.

In particular, we get that

$$|f''(x)| \leqslant 2(0.05), \quad \text{for } |x| \leqslant 0.05,$$

or

$$|f''(x)| \leqslant 0.1, \quad \text{for } |x| \leqslant 0.05.$$

Thus, we can take $M = 0.1$ in (40), with $a = 0$, to get that

$$|E_1(x;0)| \leqslant (0.05)|x|^2, \quad \text{for } |x| \leqslant 0.05.$$

In particular, since 0.047 is in the interval $[-0.05, 0.05]$, we obtain that the error in the approximation in (39) is at most

$$|E_1(0.047;0)| \leqslant (0.05)^3 = 0.000125.$$

Hence, the estimate in (39) is accurate to three decimal places. □

6. Give the equilibrium points of the differential equation $\dfrac{dy}{dt} = (y+1)^2(y-1)$ and determine the nature of their stability. Sketch possible solutions.

***Solution**:* Put $f(y) = (y+1)^2(y-1)$ for all $y \in \mathbb{R}$; then, the differential equation

$$\frac{dy}{dt} = f(y) \tag{41}$$

has equilibrium points

$$\overline{y}_1 = -1 \quad \text{and} \quad \overline{y}_2 = 1. \tag{42}$$

To determine the whether the equilibrium points in (42) are stable or not, we apply the Principle of Linearized Stability in conjunction with the sketch of the graph of f as a function of y shown in Figure 1. Note from the graph of f as a function of y that $f'(1) > 0$, since f is increasing at 1; thus, by the principle of linearized stability, $\overline{y}_2 = 1$ is unstable. On the other hand, $f'(-1) = 0$; so that, the Principle of Linearized Stability does not apply at $\overline{y}_1 = -1$.

To determine whether $\overline{y}_1 = -1$ is stable or not, we sketch a few possible solutions. We use the qualitative information given by the differential equation in (41) and the sketch of the graph of f versus y given in Figure 1.

First, observe that solutions of (41) increase for $y > 1$, and decrease for $-1 < y < 1$ and for $y < -1$. At the equilibrium points $\overline{y}_1 = -1$ and $\overline{y}_2 = 1$, the solutions remain constant. These are sketched in Figure 2. Observe also that

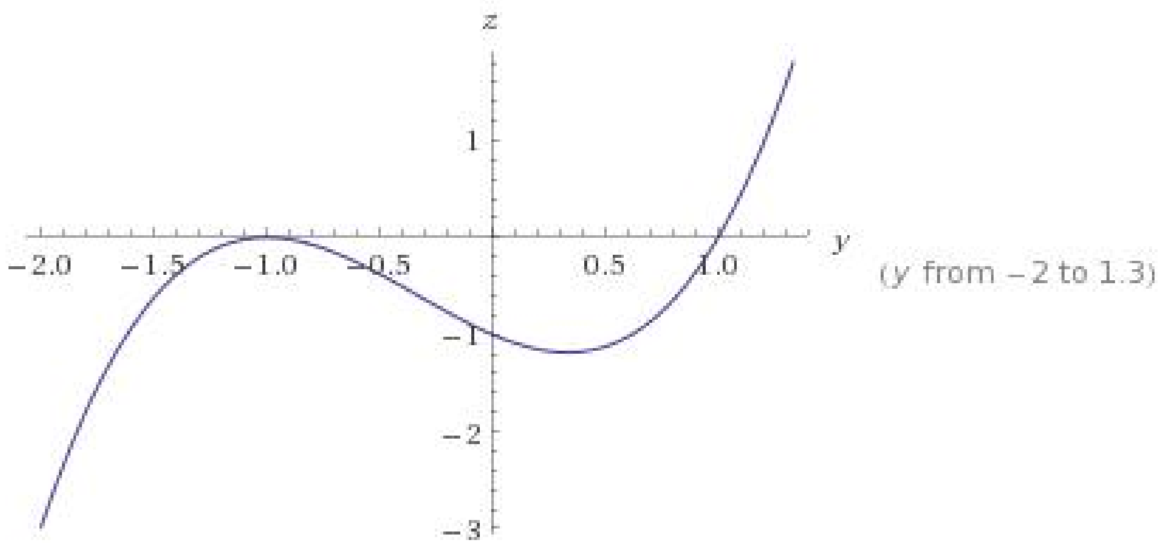

Figure 1: Sketch of Plot of f versus y

solutions of the differential equation in (41) that begin near $\overline{y}_1 = -1$, but below -1 will tend away from $\overline{y}_1 = -1$, since solutions of (41) decrease for $y < -1$. This is shown in Figure 2. Hence, $\overline{y}_1 = -1$ is unstable.

To complete the sketch of the graphs of other solutions of (41) consider the signs of the second derivative of the solutions shown in Table 1. The signs in Table

$f(y)$:	$-$	$-$	$-$	$+$
$f'(y)$:	$+$	$-$	$+$	$+$
	-1	$1/3$	1	y
y'':	$-$	$+$	$-$	$+$
Concavity:	down	up	down	up

Table 1: Concavity of the graph of y

1 were obtained from the expression for the second derivative $y''(t)$ of solutions

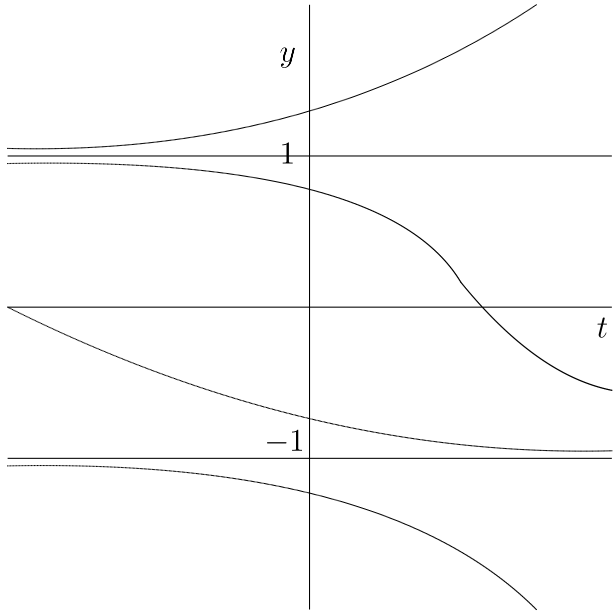

Figure 2: Sketch of graphs of possible solutions of (41)

of the equation in (41):

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{d}{dt}[f(y)] \\ &= f'(y)\frac{dy}{dt} \\ &= f'(y)f(y), \end{aligned}$$

where we have used (41) and the Chain Rule.

Note that the first derivative of f as a function of y is

$$f'(y) = (y+1)(3y-1);$$

thus, another point of interest shown in Table 1 is $y = 1/3$. This corresponds to points where the graph of solutions of (41) has an inflection point. One of the solutions sketched in Figure 2 shows this feature.

□