Review Problems for Exam 3

(1) Assume that the random variable X has mgf

$$\psi_X(t) = \frac{e^t}{4 - 3e^t}, \qquad \text{for } t < \ln\left(\frac{4}{3}\right).$$

Compute the expected value, second moment and variance of X.

(2) Let X have mgf given by

$$\psi_X(t) = \frac{1}{3}e^t + \frac{2}{3}e^{2t}, \qquad \text{for } t \in \mathbf{R}.$$

(a) Give the distribution of X

(b) Compute the expected value and variance of X.

(3) Let X have mgf given by

$$f_X(x) = \begin{cases} \dfrac{e^t - e^{-t}}{2t}, & \text{if } t \neq 0; \\ 1, & \text{if } t = 0, \end{cases}$$

(a) Give the distribution of X

(b) Compute the expected value and variance of X.

(4) A random point (X, Y) is distributed uniformly on the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

(a) Give the joint pdf for X and Y.

(b) Compute the following probabilities:
(i) $\Pr(X^2 + Y^2 < 1)$,
(ii) $\Pr(2X - Y > 0)$,
(iii) $\Pr(|X + Y| < 2)$.

(5) The random pair (X, Y) has the joint distribution

X \ Y	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

(a) Show that X and Y are not independent.

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

(6) An experiment consists of independent tosses of a fair coin. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses. Are X and Y independent random variables?

(7) Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that

$$\int_0^\infty g(t)\ dt = 1.$$

Define

$$f(x,y) = \begin{cases} \dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} & \text{for } 0 < x < \infty,\ 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

(8) Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?

(9) Assume that the number of calls coming per minute into a hotel's reservation center follows a Poisson distribution with mean 3.
(a) Find the probability that no calls come in a given 1 minute period.
(b) Assume that the number of calls arriving in two different minutes are independent. Find the probability that at least two calls will arrive in a given two minute period.

(10) Let $Y \sim \text{Binomial}(100, 1/2)$. Use the Central Limit Theorem to estimate the value of $\Pr(Y = 50)$.
Suggestion: Observe that $\Pr(Y = 50) = \Pr(49.5 < Y \leq 50.5)$, since Y is discrete.

(11) Roll a balanced die 36 times. Let Y denote the sum of the outcomes in each of the 36 rolls. Estimate the probability that $108 \leq Y \leq 144$.
Suggestion: Since the event of interest is $(Y \in \{108, 109, \ldots, 144\})$, rewrite

$$\Pr(108 \leq Y \leq 144) \text{ as } \Pr(107.5 < Y \leq 144.5).$$

(12) Forty nine digits are chosen at random and with replacement from $\{0, 1, 2, \ldots, 9\}$. Estimate the probability that their average lies between 4 and 6.

(13) Let $X_1, X_2, \ldots, X_{30}$ be independent random variables each having a discrete distribution with pmf: $p(x) = 1/4$, if $x = 0$ or $x = 2$; $p(x) = 1/2$, if $x = 1$; $p(x) = 0$ elsewhere.
Estimate the probability that $X_1 + X_2 + \cdots + X_{30}$ is at most 33.