Assignment #1

Due on Friday, September 8, 2017

Read Chapter 2 on *Variational Problems,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a differentiable function. Prove that if f attains a local maximum or minimum at some point $u_o \in U$, then $\nabla f(u_o) = 0$, where ∇f denotes the gradient of f.

2. A set $K \subseteq \mathbb{R}^n$ is said to be convex iff for any two points x, y in K, the line segment from x to y is contained in K. Let U denote an open and convex subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a C^1 function. Let $u, v \in U$ and put $g(t) = f(tv + (1-t)u)$ for all $t \in [0, 1]$. Explain why $g : [0, 1] \to \mathbb{R}$ is well defined. Show that g is differentiable in $(0, 1)$ and compute $g'(t)$ for all $t \in (0, 1)$. What is $g'(0)$?

3. Let $\mathcal{F}[a, b]$ denote the set of all real valued functions defined on the closed interval $[a, b]$. Verify that $\mathcal{F}[a, b]$ is a vector space (or linear space) under the operations of pointwise addition and scalar multiplication.

4. Let $C[a, b]$ denote the set of functions $y\colon [a, b] \to \mathbb{R}$ that are continuous on $[a, b]$.
   (a) Show that $C[a, b]$ is a subspace of $\mathcal{F}[a, b]$.
   (b) Let $C_o[a, b] = \{y \in C[a, b] \mid y(a) = y(b) = 0\}$. Show that $C_o[a, b]$ is a subspace of $C[a, b]$.
   (c) Let $C^1[a, b]$ denote the set of functions $y\colon [a, b] \to \mathbb{R}$ which are continuous on $[a, b]$, differentiable on (a, b), and such that the derivatives y' are continuous on (a, b). Show that $C^1[a, b]$ is a subspace of $C[a, b]$.

5. Verify that the following define norms in $C[a, b]$.
   (a) $||y||_1 = \displaystyle\int_a^b |y(t)|\ \mathrm{d}t \quad$ for all $y \in C[a, b]$.
   (b) $||y||_2 = \left(\displaystyle\int_a^b |y(t)|^2\ \mathrm{d}t\right)^{1/2} \quad$ for all $y \in C[a, b]$.