Assignment #2

Due on Friday, September 15, 2017

Read Section 3.1, *Geodesics in the Plane,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Sections 2–12, 2–13, 2–14, 3–1, 3–2 and 3–3, pp. 12–24, in *Calculus of Variations* by Robert Weinstock.

Background and Definitions

- **Continuity**. A function $f\colon [a,b] \to \mathbb{R}$ is said to be continuous at $x_o \in [a,b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ (which depends on ε and x_o) such that $|f(x) - f(x_o)| < \varepsilon$ for all $x \in [a,b]$ with $|x - x_o| < \delta$.
- **The Class** $C([a,b],\mathbb{R})$. If f is continuous at every point in $[a,b]$, we say that f is continuous on $[a,b]$ and write $f \in C([a,b],\mathbb{R})$.
- **The Class** $C_o([a,b],\mathbb{R})$. If f is continuous at every point in $[a,b]$ and $f(a) = 0$ and $f(b) = 0$, we write $f \in C_o([a,b],\mathbb{R})$.
- **The Class** $C^1([a,b],\mathbb{R})$. If f is differentiable in an open interval that contains $[a,b]$, and f' is continuous on $[a,b]$, we write $f \in C^1([a,b],\mathbb{R})$.
- **The Class** $C_o^1([a,b],\mathbb{R})$. If $f \in C^1([a,b],\mathbb{R})$ and $f(a) = f(b) = 0$, we write $f \in C_o^1([a,b],\mathbb{R})$.

Do the following problems

1. Prove that if $f \in C[a,b]$ and $f(x_o) \neq 0$ for some $x_o \in (a,b)$, then there exists an interval $(x_o - \delta, x_o + \delta)$ contained in (a,b) such that $f(x) \neq 0$ for all $x \in (x_o - \delta, x_o + \delta)$.

2. Assume that $f \in C([a,b],\mathbb{R})$ and that $f(x) \geqslant 0$ for all $x \in [0,1]$. Prove that, if

$$\int_a^b f(x)\, dx = 0,$$

then $f(x) = 0$ for all $x \in [a,b]$.

3. Assume that $f \in C([a,b],\mathbb{R})$. Suppose that

$$\int_c^d f(x)\ dx = 0,$$

for all c and d such that $a \leqslant c < d \leqslant b$. Show that $f(x) = 0$ for all $x \in [a,b]$.

4. **The Fundamental Lemma in the Calculus of Variations**. Let $f \in C([a,b],\mathbb{R})$ and suppose that

$$\int_a^b f(x)\eta(x)\ dx = 0, \qquad \text{for all } \eta \in C_o([a,b],\mathbb{R}).$$

Show that $f(x) = 0$ for all $x \in [a,b]$.

5. **The Second Fundamental Lemma in the Calculus of Variations**. In this problem we prove the second fundamental lemma in the Calculus of Variations: Let $f \in C([a,b],\mathbb{R})$ and suppose that

$$\int_a^b f(x)\eta'(x)\ dx = 0, \qquad \text{for all } \eta \in C_o^1([a,b],\mathbb{R}).$$

Then, f must be constant on $[a,b]$.

(a) Put

$$c = \frac{1}{b-a}\int_a^b f(x)\ dx$$

and define $\eta\colon [a,b] \to \mathbb{R}$ by

$$\eta(x) = \int_a^x (f(t) - c)\ dt, \qquad \text{for } x \in [a,b].$$

Verify that $\eta \in C_o^1([a,b],\mathbb{R})$.

(b) Show that

$$\int_a^b (f(x) - c)^2\ \mathrm{d}x = 0.$$

(c) Deduce that $f(x) = c$ for all $x \in [a,b]$.