Assignment #7

Due on Friday, November 10, 2017

Read Chapter 5, *Optimization Problems with Constraints,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Chapter 4, *Isoperimetric Problems,* in *Calculus of Variations* by Robert Weinstock.

Do the following problems

1. Let $V = C^1([a,b],\mathbb{R}^2)$; so that, the elements of V are vector–valued functions

$$(x,y)\colon [a,b] \to \mathbb{R}^2,$$

whose values are denoted by $(x(t), y(t))$, for $t \in [a,b]$, where the functions $x\colon [a,b] \to \mathbb{R}$ and $y\colon [a,b] \to \mathbb{R}$ are differentiable functions of $t \in (a,b)$, with continuous derivatives $\dot{x}$ and $\dot{y}$ (the dot on top of a variable name indicates derivative with respect to t). We denote by V_o the space of vector valued functions $(\eta_1,\eta_2) \in V$ such that

$$\eta_1(a) = \eta_1(b) = \eta_2(a) = \eta_2(b) = 0.$$

   (a) Show that V is a vector space.

   (b) Show that V_o is a subspace of V

2. Let V be as in Problem 1. For $(x,y) \in V$, define

$$\|(x,y)\| = \max_{a\leqslant t\leqslant b} |x(t)| + \max_{a\leqslant t\leqslant b} |y(t)| + \max_{a\leqslant t\leqslant b} |\dot{x}(t)| + \max_{a\leqslant t\leqslant b} |\dot{y}(t)|.$$

Verify that $\|(\cdot,\cdot)\|$ defines a norm in V.

3. Let V and V_o be as in Problem 1. Consider a function $F\colon [a,b]\times\mathbb{R}^4 \to \mathbb{R}$ whose values are denoted by $F(t,x,y,p,q)$ for $t \in [a,b]$ and real variables x, y, p and q. We assume that the F has partial derivatives

$$F_x(t,x,y,p,q),\ F_y(t,x,y,p,q),\ F_p(t,x,y,p,q) \text{ and } F_q(t,x,y,p,q),$$

which are assumed to be continuous on $[a,b]\times\mathbb{R}^4$.

Define the functional $J\colon V \to \mathbb{R}$ by

$$J((x,y)) = \int_a^b F(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\,dt, \qquad \text{for all } (x,y) \in V. \tag{1}$$

For $(x,y) \in V$ and $(\eta_1,\eta_2) \in V_o$, define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(s) = J((x,y) + s((\eta_1,\eta_2)) = J((x+s\eta_1, y+s\eta_2)), \qquad \text{for all } s \in \mathbb{R}.$$

(a) Show that $g\colon \mathbb{R} \to \mathbb{R}$ is differentiable and compute $g'(s)$ for all $s \in \mathbb{R}$.

(b) Deduce that $J\colon V \to \mathbb{R}$ is Gâteaux differentiable at every $(x, y) \in V$ in the direction of $(\eta_1, \eta_2) \in V_o$, and compute $dJ((x, y); (\eta_1, \eta_2))$.

4. Let V and V_o be as in Problem 1 and $J\colon V \to \mathbb{R}$ as in Problem 3. Define the set

$$\mathcal{A} = \{(x, y) \in V \mid x(a) = x_o,\ x(b) = x_1, y(a) = y_o,\ \text{and } y(b) = y_1\},$$

where x_o, x_1, y_o and y_1 are given real numbers.

Assume that $(x, y) \in \mathcal{A}$ is an optimizer of J over the class $\mathcal{A}$.

(a) Show that

$$dJ((x, y); (\eta_1, \eta_2)) = 0, \qquad \text{for all } (\eta_1, \eta_2) \in V_o.$$

(b) Show that (x, y) must be solution of the system of equations

$$\begin{cases} \dfrac{d}{dt}[F_p(t, x, y, \dot{x}, \dot{y})] & = & F_x(t, x, y, \dot{x}, \dot{y}); \\ & & \\ \dfrac{d}{dt}[F_q(t, x, y, \dot{x}, \dot{y})] & = & F_y(t, x, y, \dot{x}, \dot{y}). \end{cases}$$

5. Let $V = C^1([0, 1], \mathbb{R}^2)$, $V_o = C^1_o([0, 1], \mathbb{R}^2)$ and

$$\mathcal{A} = \{(x, y) \in V \mid x(0) = x_o,\ x(1) = x_1, y(0) = y_o,\ \text{and } y(1) = y_1\},$$

where x_o, x_1, y_o and y_1 are given real numbers.

Define the functional $J\colon V \to \mathbb{R}$ by

$$J((x, y)) = \int_0^1 \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\ dt, \qquad \text{for all } (x, y) \in V.$$

Consider the optimization problem: Find $(x, y) \in \mathcal{A}$ such that

$$J((x, y)) \leqslant J(u, v), \qquad \text{for all } (u, v) \in \mathcal{A}$$

(a) Give necessary conditions for $(x, y) \in \mathcal{A}$ to be a solution of the optimization problem.

(b) Give a candidate $(x, y) \in \mathcal{A}$ for a solution of the optimization problem. Give a geometric interpretation of your result.