Assignment #8

Due on Friday, November 17, 2017

Read Section 1.6.1 on *Divergence Theorem*, pp. 46–57, in *Introduction to Partial Differential Equations and Hilbert Space Methods* by Karl E. Gustafson.

Background and Definitions

Divergence. Let U be an open subset of $\mathbb{R}^2$ and $\overrightarrow{F} \in C^1(U, \mathbb{R}^2)$ be a vector field given by

$$\overrightarrow{F}(x, y) = (P(x, y), Q(x, y)), \quad \text{for } (x, y) \in U,$$

where $P \in C^1(U, \mathbb{R})$ and $Q \in C^1(U\,\mathbb{R})$ are C^1, real–valued functions defined on U. The divergence of $\overrightarrow{F}$, denoted $\operatorname{div}\overrightarrow{F}$, is the scalar field, $\operatorname{div}\overrightarrow{F} : U \to \mathbb{R}$ defined by

$$\operatorname{div}\overrightarrow{F}(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)), \quad \text{for } (x, y) \in U.$$

Gradient. Let U be an open subset of $\mathbb{R}^2$ and $u \in C^1(U, \mathbb{R})$ be a scalar field. The gradient of u, denoted ∇u, is the vector field, $\nabla u \colon U \to \mathbb{R}^2$ defined by

$$\nabla u(x, y) = \left(\frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right), \quad \text{for } (x, y) \in U.$$

Laplacian. Let U be an open subset of $\mathbb{R}^2$ and $u \in C^2(U, \mathbb{R})$ be a scalar field. The divergence of the gradient of u, $\operatorname{div}\nabla u$, is called the Laplacian of u, denoted by Δu. Thus,

$$\Delta u = \operatorname{div}\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The Divergence Theorem in $\mathbb{R}^2$. Let U be an open subset of $\mathbb{R}^2$ and Ω an open subset of U such that $\overline{\Omega} \subset U$. Suppose that Ω is bounded with boundary $\partial\Omega$. Assume that $\partial\Omega$ is a piece–wise C^1, simple, closed curve. Let $\overrightarrow{F} \in C^1(U, \mathbb{R}^2)$. Then,

$$\iint_\Omega \operatorname{div}\overrightarrow{F}\; dxdy = \oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\; ds, \tag{1}$$

where $\widehat{n}$ is the outward, unit, normal vector to $\partial\Omega$ that exists everywhere on $\partial\Omega$, except possibly at finitely many points.

Do the following problems

1. Let U be an open subset of $\mathbb{R}^2$, $\overrightarrow{F} \in C^1(U, \mathbb{R}^2)$ be a vector field and $u \in C^1(U, \mathbb{R})$ be a scalar field. Show that

   $$\operatorname{div}(u\overrightarrow{F}) = \nabla u \cdot \overrightarrow{F} + u\, \operatorname{div}\overrightarrow{F},$$

   where $\nabla u \cdot \overrightarrow{F}$ denotes the dot–product of ∇u and $\overrightarrow{F}$.

2. Let U be an open subset of $\mathbb{R}^2$, $u \in C^2(U,\mathbb{R})$ and $v \in C^1(U,\mathbb{R})$. Show that

$$\text{div}(v\nabla u) = \nabla v \cdot \nabla u + v\ \Delta u,$$

where $\nabla v \cdot \nabla u$ denotes the dot–product of ∇v and ∇u, and Δu is the Laplacian of u.

3. Let U be an open subset of $\mathbb{R}^2$ and Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subset U$. Assume that the boundary, $\partial\Omega$, of Ω is a simple closed curve parametrized by $\sigma \in C^1([0,1],\mathbb{R}^2)$. Let $u \in C^2(U,\mathbb{R})$ and $v \in C^1(U,\mathbb{R})$. Apply the Divergence Theorem (1) to the vector field $\overrightarrow{F} = v\nabla u$ to obtain

$$\iint_\Omega \nabla u \cdot \nabla v\ dxdy + \iint_\Omega v\Delta u\ dxdy = \oint_{\partial\Omega} v\frac{\partial u}{\partial n}\ ds, \qquad (2)$$

where Δu is the Laplacian of u and $\dfrac{\partial u}{\partial n}$ is the directional derivative of u in the direction of a unit vector perpendicular to $\partial\Omega$ which points away from Ω. This is usually referred to as **Green's identity I** (see p. 47 in Gustafson's book).

4. Let U be an open subset of $\mathbb{R}^2$ and Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subset U$. Assume that the boundary, $\partial\Omega$, of Ω is a simple closed curve parametrized by $\sigma \in C^1([0,1],\mathbb{R}^2)$. Put

$$C_o^1(\Omega,\mathbb{R}) = \{v \in C^1(U,\mathbb{R}) \mid v = 0 \text{ on } \partial\Omega\};$$

that is, $C_o^1(\Omega,\mathbb{R})$ is the space of C^1 functions in Ω that vanish on the boundary of Ω. Let $u \in C^2(U,\mathbb{R})$. Use Green's identity I in (2) to show that

$$\iint_\Omega \nabla v \cdot \nabla u\ dxdy = -\iint_\Omega v\Delta u\ dxdy, \qquad \text{for all } v \in C_o^1(\Omega,\mathbb{R}).$$

5. Let U and Ω be as in Problem 4. A function $u \in C^2(U,\mathbb{R})$ is said to satisfy Laplace's equation in Ω if

$$\Delta u(x,y) = 0, \qquad \text{for all } (x,y) \in \Omega. \qquad (3)$$

A function $u \in C^2(U,\mathbb{R})$ satisfying (3) is also said to be *harmonic* in Ω.

(a) Use the result from Problem 4 to show that, for any $u \in C^2(U,\mathbb{R})$ that is harmonic in Ω,

$$\iint_\Omega \nabla u \cdot \nabla v\ dxdy = 0, \qquad \text{for all } v \in C_o^1(\Omega,\mathbb{R}).$$

(b) Assume that $u \in C^2(U,\mathbb{R})$ is harmonic in Ω. Show that, if $u = 0$ on $\partial\Omega$, then $u(x,y) = 0$ for all $(x,y) \in \Omega$.