Review Problems for Exam 1

1. Let Ω be a bounded region in the xy–plane, and let $G = G(x, y)$ be continuous function defined in Ω. Suppose that

$$\int\int_{\Omega} G(x, y)v(x, y) \text{ d}x \text{ d}y = 0$$

for every continuously differentiable function v defined in Ω and vanishing on the boundary of Ω. Show that $G(x, y) = 0$ for every $(x, y) \in \Omega$.

2. Let $f \in C([a, b], \mathbb{R})$ and $g \in C([a, b], \mathbb{R})$ be given. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \int_a^b [f(x) - tg(x)]^2 \; dx, \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

Observe that $h(t) \geqslant 0$ for all $t \in \mathbb{R}$. Observe also that h is a differentiable function of t.

(a) Assume that $\displaystyle\int_a^b (g(x))^2 \; dx \neq 0$. Find the value $t_o \in \mathbb{R}$ such that

$$h(t_o) \leqslant h(t), \qquad \text{for all } t \in \mathbb{R}.$$

(b) Assume that $\displaystyle\int_a^b (g(x))^2 \; dx \neq 0$, and let t_o be as in the previous part. Use the observation $h(t_o) \geqslant 0$ to the deduce the inequality

$$\left(\int_a^b f(x)g(x) \; dx\right)^2 \leqslant \int_a^b (f(x))^2 \; dx \cdot \int_a^b (g(x))^2 \; dx.$$

This leads to the Cauchy–Schwarz inequality

$$\left|\int_a^b f(x)g(x) \; dx\right| \leqslant \sqrt{\int_a^b (f(x))^2 \; dx} \cdot \sqrt{\int_a^b (g(x))^2 \; dx}.$$

(c) When does the inequality in part (b) yields equality?

3. Let $V = \left\{ y \in C^1([0, 1], \mathbb{R}) \;\Big|\; \displaystyle\int_0^1 (y'(x))^2 \text{ d}x < \infty \right\}$, and define $J\colon V \to \mathbb{R}$ by

$$J(y) = \frac{1}{2}\int_0^1 (y'(x))^2 \text{ d}x \quad \text{for all } y \in V.$$

(a) Prove that J is Gâteaux differentiable and compute $dJ(y;v)$ for $y, v \in V$.

(b) Prove that J is convex but not strictly convex.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real valued function of a single variable, and assume that f is twice differentiable with continuous second derivative $f''\colon \mathbb{R} \to \mathbb{R}$. Let $V = C^1([a,b],\mathbb{R})$ and define $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b f(y'(x))\, dx, \qquad \text{for all } y \in V.$$

Put $V_o = C^1_o([a,b],\mathbb{R})$ and define

$$\mathcal{A} = \{y \in C^1[a,b] \mid y(a) = y_o \ \text{ and } \ y(b) = y_1\}$$

for given real numbers y_o and y_1.

(a) Show that if $f''(z) > 0$ for all $z \in \mathbb{R}$, then J is strictly convex in $\mathcal{A}$.

(b) Give the Euler–Lagrange equation associated with J and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

(c) Find the unique minimizer of J in $\mathcal{A}$.

5. Let $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote a continuous function of two variables y and z. Let $V = C^1([a,b],\mathbb{R})$ and define the functional

$$J(y) = \int_a^b F(y(x), y'(x))\, \mathrm{d}x, \quad \text{for } y \in V. \tag{2}$$

(a) Assume that $y(a) = y_o$ and $y(b) = y_1$ and assume that $y_o < y_1$. Make a change of variables to express the functional in (2) in terms of an integral with respect to y of the form

$$J(x) = \int_{y_o}^{y_1} G(y, x'(y))\, \mathrm{d}y, \quad \text{for } x \in C^1([y_o, y_1],\mathbb{R}). \tag{3}$$

Express the function G in terms of F.

(b) Derive the Euler–Lagrange equation associated with the functional J given in (3).

(c) Solve the differential equation derived in the previous part and deduce that, if y is a solution of the Euler–Lagrange equation associated with the functional J given in (2), then

$$y'F_z(y, y') - F(y, y') = C,$$

for some constant C.

6. Let V denote a normed, linear space, and V_o a nontrivial subspace of V, and Assume that $J\colon V \to \mathbb{R}$ and $J_1\colon V \to \mathbb{R}$ are Gâteaux differentiable in V along any direction $v \in V_o$.

    (a) Show that $dJ(u; cv) = c\, dJ(u; v)$ for any $c \in \mathbb{R}$.

    (b) Show that $J + J_1$ is Gâteaux differentiable at any $u \in V$ in the direction of $v \in V_o$, and

$$d(J + J_1)(u; v) = dJ(u; v) + dJ_1(u; v).$$

7. Let V denote a normed, linear space, and V_o a nontrivial subspace of V, and $\mathcal{A}$ a nonempty subset of V. Assume that $J_1\colon V \to \mathbb{R}$ and $J_2\colon V \to \mathbb{R}$ are Gâteaux differentiable in V along any direction $v \in V_o$.

    (a) Show that if J_1 and J_2 are convex in $\mathcal{A}$, then so are $c^2 J_1$ and $J_1 + J_2$, for any $c \in \mathbb{R}$.

    (b) Show that if J_1 is convex in $\mathcal{A}$ and J_2 is strictly convex in $\mathcal{A}$, then $J_1 + J_2$ is strictly convex in $\mathcal{A}$.

8. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_0^{1/2} [y(x) + \sqrt{1 + (y'(x))^2}]\ \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1[0, 1/2] \mid y(0) = -1 \text{ and } y(1/2) = -\sqrt{3}/2\}$. Verify that J is strictly convex and find, if possible, the unique minimizing function for J in $\mathcal{A}$.

9. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_1^2 [y(x) + xy'(x)]\ \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1[1, 2] \mid y(1) = 1 \text{ and } y(2) = 2\}$. Verify that J is convex, but not strictly convex in $\mathcal{A}$. Can you find more than one function which minimizes J in $\mathcal{A}$?