Solutions to Review Problems for Exam #1

1. Let Ω be a bounded region in the xy–plane, and let $G = G(x, y)$ be continuous function defined in Ω. Suppose that

$$\iint_{\Omega} G(x, y)v(x, y)\mathrm{d}x\mathrm{d}y = 0$$

for every continuously differentiable function v defined in Ω and vanishing on the boundary of Ω. Show that $G(x, y) = 0$ for every $(x, y) \in \Omega$.

Solution: Suppose that

$$\iint_{\Omega} G(x, y)v(x, y)\mathrm{d}x\mathrm{d}y = 0, \quad \text{for all } v \in C_o(\Omega, \mathbb{R}). \tag{1}$$

Assume, by way of contradiction, that there exists $(x_o, y_o) \in \Omega$ such that $G(x_o, y_o) \neq 0$. We may assume without loss of generality that $G(x_o, y_o) > 0$.

Using the continuity of G, with $\varepsilon = \dfrac{G(x_o, y_o)}{2}$, and the assumption that Ω is open, we obtain $\delta > 0$ such that the open rectangle

$$R_\delta(x_o, y_o) = (x_o - \delta, x_o + \delta) \times (y_o - \delta, y_o + \delta)$$

is contained in Ω, and

$$G(x, y) > \frac{G(x_o, y_o)}{2} > 0, \quad \text{for all } (x, y) \in R_\delta, \tag{2}$$

where we have used R_δ to denote $R_\delta(x_o, y_o)$.

Define $v\colon \Omega \to \mathbb{R}$ as follows:

$$v(x, y) = \begin{cases} (x - x_o + \delta)(x_o + \delta - x)(y - y_o + \delta)(y_o + \delta - y), & \text{if } (x, y) \in R_\delta \\ 0, & \text{if } (x, y) \in \Omega \backslash R_\delta. \end{cases}$$

Consequently, $v \in C_o(\Omega, \mathbb{R})$; furthermore, $v(x, y) = 0$ for $(x, y) \notin R_\delta$, and

$$v(x, y) > 0, \quad \text{for all } (x, y) \in R_\delta. \tag{3}$$

It follows from the definition of v and the estimates in (2) and (3) that

$$\iint_{\Omega} G(x, y)v(x, y)\ dxdy = \iint_{R_\delta} G(x, y)v(x, y)\ dxdy > 0,$$

which is in direct contradiction with the assumption in (1). We therefore conclude that $G(x, y) = 0$ for all $(x, y) \in \Omega$, which was to be shown. □

2. Let $f \in C([a,b],\mathbb{R})$ and $g \in C([a,b],\mathbb{R})$ be given. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \int_a^b [f(x) - tg(x)]^2\ dx, \qquad \text{for all } t \in \mathbb{R}. \tag{4}$$

Observe that $h(t) \geqslant 0$ for all $t \in \mathbb{R}$. Observe also that h is a differentiable function of t.

(a) Assume that $\displaystyle\int_a^b (g(x))^2\ dx \neq 0$. Find the value $t_o \in \mathbb{R}$ such that

$$h(t_o) \leqslant h(t), \qquad \text{for all } t \in \mathbb{R}. \tag{5}$$

***Solution**:* Expand the integrand in (4) to compute

$$h(t) = \int_a^b (f(x))^2\ dx - 2t\int_a^b f(x)g(x)\ dx + t^2\int_a^b (g(x))^2\ dx, \tag{6}$$

for $t \in \mathbb{R}$. We then see that h is a quadratic polynomial. It is therefore differentiable with derivative

$$h'(t) = -2\int_a^b f(x)g(x)\ dx + 2t\int_a^b (g(x))^2\ dx, \qquad \text{for } t \in \mathbb{R}, \tag{7}$$

and second derivative

$$h''(t) = 2\int_a^b (g(x))^2\ dx, \qquad \text{for } t \in \mathbb{R}. \tag{8}$$

Since we are assuming that $\displaystyle\int_a^b (g(x))^2\ dx \neq 0$, we obtain from (8) that $h''(t) > 0$ for all $t \in \mathbb{R}$. Thus, h has a unique minimizer at a point t_o such that $h'(t_o) = 0$; or, according to (7), at

$$t_o = \frac{\displaystyle\int_a^b f(x)g(x)\ dx}{\displaystyle\int_a^b (g(x))^2\ dx}. \tag{9}$$

For this value of t, the estimate in (5) holds true. □

(b) Assume that $\displaystyle\int_a^b (g(x))^2\ dx \neq 0$, and let t_o be as in the previous part. Use the observation $h(t_o) \geqslant 0$ to the deduce the inequality

$$\left(\int_a^b f(x)g(x)\ dx\right)^2 \leqslant \int_a^b (f(x))^2\ dx \cdot \int_a^b (g(x))^2\ dx. \tag{10}$$

This leads to the Cauchy–Schwarz inequality

$$\left|\int_a^b f(x)g(x)\ dx\right| \leqslant \sqrt{\int_a^b (f(x))^2\ dx} \cdot \sqrt{\int_a^b (g(x))^2\ dx}. \tag{11}$$

***Solution**:* From the definition of $h(t)$ in (4) we see that

$$h(t) \geqslant 0, \quad \text{for all } t \in \mathbb{R}.$$

Thus, in particular,

$$h(t_o) \geqslant 0, \tag{12}$$

or, in view of (6),

$$\int_a^b (f(x))^2\ dx - 2t_o \int_a^b f(x)g(x)\ dx + t_o^2 \int_a^b (g(x))^2\ dx \geqslant 0. \tag{13}$$

Next, substitute the value of t_o in (9) into the estimate in (13) to obtain

$$\int_a^b (f(x))^2\ dx - 2\frac{\left(\int_a^b f(x)g(x)\ dx\right)^2}{\int_a^b (g(x))^2\ dx} + \frac{\left(\int_a^b f(x)g(x)\ dx\right)^2}{\int_a^b (g(x))^2\ dx} \geqslant 0,$$

or

$$\int_a^b (f(x))^2\ dx - \frac{\left(\int_a^b f(x)g(x)\ dx\right)^2}{\int_a^b (g(x))^2\ dx} \geqslant 0,$$

from which we get that

$$\left(\int_a^b f(x)g(x)\ dx\right)^2 \leqslant \int_a^b (f(x))^2\ dx \int_a^b (g(x))^2\ dx,$$

which is (10). Taking the positive square root on both sides of (10) then yields (11). □

(c) When does the inequality in part (b) yields equality?

Solution: Equality in the inequality in (10) occurs when equality in (12) occurs; namely,

$$h(t_o) = 0,$$

or, using the definition of h in (4),

$$\int_a^b [f(x) - t_o g(x)]^2 \; dx = 0. \tag{14}$$

Consequently, since f and g are assumed to be continuous on $[a, b]$, it follows from (14) that

$$f(x) = t_o g(x), \qquad \text{for all } x \in [a, b].$$

Hence, equality in the Cauchy–Schwarz inequality occurs if and only if f is a constant multiple of g. □

3. Let $V = \left\{ y \in C^1([0,1], \mathbb{R}) \;\middle|\; \int_0^1 (y'(x))^2 \mathrm{d}x < \infty \right\}$, and define $J\colon V \to \mathbb{R}$ by

$$J(y) = \frac{1}{2}\int_0^1 (y'(x))^2 \mathrm{d}x \quad \text{for all } y \in V. \tag{15}$$

(a) Prove that J is Gâteaux differentiable and compute $dJ(y; v)$ for $y, v \in V$.

Solution: For y and v in V, and $t \in \mathbb{R}$, compute

$$\begin{aligned} J(y + tv) &= \frac{1}{2}\int_0^1 (y'(x) + tv'(x))^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^1 (y'(x))^2 \mathrm{d}x + t\int_0^1 y'(x)v'(x)\mathrm{d}x + \frac{t^2}{2}\int_0^1 (v'(x))^2 \mathrm{d}x; \end{aligned}$$

so that $J(y + tv)$ is a quadratic polynomial in t. Hence, the map

$$t \mapsto J(y + tv), \qquad \text{for } t \in \mathbb{R},$$

is differentiable with derivative

$$\frac{d}{dt}[J(y + tv)] = \int_0^1 y'(x)v'(x)\mathrm{d}x + t\int_0^1 (v'(x))^2 \mathrm{d}x, \qquad \text{for all } t \in \mathbb{R}.$$

Consequently, J is Gâteaux differentiable at every $y \in V$ in the direction of v in V, with Gâteaux derivative

$$dJ(y;v) = \frac{d}{dt}[J(y+tv)]\Big|_{t=0} = \int_0^1 y'(x)v'(x)\mathrm{d}x, \tag{16}$$

for $y, v \in V$.

□

(b) Prove that J is convex but not strictly convex.

Solution: For y and v in V, compute

$$\begin{aligned} J(y+v) &= \frac{1}{2}\int_0^1 (y'(x)+v'(x))^2\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 (y'(x))^2\mathrm{d}x + \int_0^1 y'(x)v'(x)\mathrm{d}x + \frac{1}{2}\int_0^1 (v'(x))^2\mathrm{d}x; \end{aligned}$$

so that, in view of (16) and the definition of J in (15),

$$J(y+v) = J(y) + dJ(y;v) + \frac{1}{2}\int_0^1 (v'(x))^2\mathrm{d}x. \tag{17}$$

Consequently,

$$J(y+v) \geqslant J(y) + dJ(y;v), \quad \text{for } y, v \in V, \tag{18}$$

which shows that J is convex in V.

Now, in view of (17), equality in (18) holds if and only if

$$\int_0^1 (v'(x))^2\mathrm{d}x = 0;$$

so that, since v' is continuous, $v'(x) = 0$ for all $x \in [0,1]$. Thus, $v(x) = c$ for all $x \in [0,1]$, where c is a constnat not necessarily 0. Hence, J is not strictly convex. □

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real valued function of a single variable, and assume that f is twice differentiable with continuous second derivative $f''\colon \mathbb{R} \to \mathbb{R}$. Let $V = C^1([a,b],\mathbb{R})$ and define $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b f(y'(x))\ dx, \quad \text{for all } y \in V. \tag{19}$$

Put $V_o = C_o^1([a,b],\mathbb{R})$ and define

$$\mathcal{A} = \{y \in C^1[a,b] \mid y(a) = y_o \ \text{ and } \ y(b) = y_1\} \tag{20}$$

for given real numbers y_o and y_1.

(a) Show that if $f''(z) > 0$ for all $z \in \mathbb{R}$, then J is strictly convex in $\mathcal{A}$.

***Solution**:* By the arguments used in Example 4.1.3 in the lecture notes, we can show that the the functional J defined in (19) is Gâteaux differentiable for every $y \in \mathcal{A}$ in the direction of $\eta \in V_o$, with Gâteaux derivative given by

$$dJ(y;\eta) = \int_a^b f'(y'(x))\eta'(x)\, dx, \qquad \text{for } y \in \mathcal{A} \text{ and } \eta \in V_o, \tag{21}$$

since $f' \colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

Since we are also assuming that the $f''(z) > 0$ for all $z \in \mathbb{R}$, the arguments used in Example 4.3.7 in the lecture notes can be used to show that

$$f(z+w) \geqslant f(z) + f'(z)w, \qquad \text{for all } z, w \in \mathbb{R}, \tag{22}$$

$$\text{with equality if and only if } w = 0. \tag{23}$$

It follows from (22), the definition of J in (19) and (21) that

$$J(y+\eta) \geqslant J(y) + dJ(y;\eta), \qquad \text{for } y \in \mathcal{A} \text{ and } \eta \in V_o, \tag{24}$$

which shows that J is convex in $\mathcal{A}$.

To show that J is strictly convex in $\mathcal{A}$, assume that equality in (24) holds true for some $y \in \mathcal{A}$ and $\eta \in V_o$. Then, using the definition of J in (19) and the result in (21),

$$\int_a^b f(y'(x) + \eta'(x))\, dx = \int_a^b f(y'(x))\, dx + \int_a^b f'(y'(x))\eta'(x)\, dx,$$

which we can rewrite as

$$\int_a^b [f(y'(x) + \eta'(x)) - f(y'(x)) - f'(y'(x))\eta'(x)]\, dx = 0. \tag{25}$$

It follows from the inequality in (22) that the integrand in (25) is nonnegative. Hence, since f, f', y' and η' are continuous on $[a,b]$, it follows from (25) that

$$f(y'(x) + \eta'(x)) - f(y'(x)) - f'(y'(x))\eta'(x) = 0, \qquad \text{for all } x \in [a,b].$$

It then follows form (22) and (23) that

$$\eta'(x) = 0, \qquad \text{for all } x \in [a, b];$$

so that, $\eta(x) = c$ for all $x \in [a, b]$, where c is a constant. Now, since $\eta \in V_o$, $\eta(a) = 0$; therefore, $c = 0$. Consequently, equality in (24) holds true if and only if $\eta(x) = 0$ for all $x \in [a, b]$. This shows that J given in (19) is strictly convex in $\mathcal{A}$. □

(b) Give the Euler–Lagrange equation associated with J and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

***Solution**:* The Euler–Lagrange equation associated with the functional J defined in (19) is

$$\frac{d}{dx}[f'(y'(x))] = 0, \qquad \text{for } x \in (a, b),$$

from which we get that

$$f'(y'(x)) = C, \qquad \text{for all } x \in [a, b], \tag{26}$$

and some constant C.

It follows from the assumptions that $f''(z) > 0$ for all $z \in \mathbb{R}$, that $f'(z)$ is a strictly increasing function of z. Consequently, we obtain from (26) that

$$y'(x) = c_1, \qquad \text{for all } x \in [a, b], \tag{27}$$

and some constant c_1.

Integrating the equation in (27) we obtain that

$$y(x) = c_1 x + c_2, \qquad \text{for all } x \in [a, b], \tag{28}$$

and some constants c_1 and c_2.

Using the assumption that $y\mathcal{A}$, we see that y must also satisfy the conditions

$$y(a) = y_o \quad \text{and} \quad y(b) = y_1, \tag{29}$$

according to the definition of $\mathcal{A}$ in (20). Applying the boundary conditions in (29) to the function in (28) we find that

$$c_1 = \frac{y_1 - y_o}{b - a} \quad \text{and} \quad c_3 = \frac{by_o - ay_1}{b - a}.$$

Thus, using the expression for y in (28),

$$y(x) = \frac{y_1 - y_o}{b - a}\, x + \frac{by_o - ay_1}{b - a}, \qquad \text{for } a \leqslant x \leqslant b. \tag{30}$$

□

(c) Find the unique minimizer of J in $\mathcal{A}$.

***Solution**:* The function in (30) is the unique minimizer of J in $\mathcal{A}$ because J is strictly convex, as shown in part (a) of this problem. This assertion follows from the convex minimization theorem proved in class and in the lecture notes (see Theorem 4.4.1). □

5. Let $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote a continuous function of two variables y and z. Let $V = C^1([a,b],\mathbb{R})$ and define the functional

$$J(y) = \int_a^b F(y(x), y'(x))\mathrm{d}x, \quad \text{for } y \in V. \tag{31}$$

(a) Assume that $y(a) = y_o$ and $y(b) = y_1$ and assume that $y_o < y_1$. Make a change of variables to express the functional in (31) in terms of an integral with respect to y of the form

$$J(x) = \int_{y_o}^{y_1} G(y, x'(y))\mathrm{d}y, \quad \text{for } x \in C^1([y_o, y_1], \mathbb{R}). \tag{32}$$

Express the function G in terms of F.

***Solution**:* Assume that $y'(x) \neq 0$ for all $x \in [a,b]$ and make a change of variables to rewrite the integral in (31) as

$$\int_a^b F(y(x), y'(x))\mathrm{d}x = \int_{y(a)}^{y(b)} F(y, y'(x))\frac{dx}{dy}\, dy,$$

where, by virtue of the Chain Rule,

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}},$$

or

$$x' = \frac{1}{y'},$$

where we have used x' to denote $x'(y)$ and y' to denote $y'(x)$. We then have that

$$\int_a^b F(y(x), y'(x))\mathrm{d}x = \int_{y_o}^{y_1} F\left(y, \frac{1}{x'}\right) x'\, dy,$$

Thus, defining

$$G(y,z) = zF\left(y, \frac{1}{z}\right), \qquad \text{for } y \in \mathbb{R} \text{ and } z \neq 0. \tag{33}$$

We then have that

$$\int_a^b F(y(x), y'(x))\mathrm{d}x = \int_{y_o}^{y_1} G(y, x')\ dy,$$

which yields the expression in (32). This expresses the functional in (31) in terms of x as a function of y for $y_o \leqslant y \leqslant y_1$. Thus, we have exchanged the roles of the variables x and y. In (32) y is thought of as the independent variable and x is the dependent variable. □

(b) Derive the Euler–Lagrange equation associated with the functional J given in (32).

***Solution**:* Thinking of y as the independent variable and of x as the dependent variable, the Euler–Lagrange equation corresponding to the functional in (32) is

$$\frac{d}{dy}[G_z(y, x')] = G_x(y, x'),$$

or

$$\frac{d}{dy}[G_z(y, x')] = 0, \tag{34}$$

since $G(y, z)$ does not depend explicitly on x. □

(c) Solve the differential equation derived in the previous part and deduce that, if y is a solution of the Euler–Lagrange equation associated with the functional J given in (31), then

$$y'F_z(y, y') - F(y, y') = C, \tag{35}$$

for some constant C.

***Solution**:* Integrate the differential equation in (34) to obtain

$$G_z(y, x') = c_1, \tag{36}$$

for some constant c_1, where, according to (33),

$$G_z(y, z) = F\left(y, \frac{1}{z}\right) + zF_z\left(y, \frac{1}{z}\right) \cdot \left(-\frac{1}{z^2}\right),$$

where we have used the Product Rule and the Chain Rule; so that,

$$G_z(y, z) = F\left(y, \frac{1}{z}\right) - \frac{1}{z}F_z\left(y, \frac{1}{z}\right). \tag{37}$$

Substituting x' for z in (37) and using the fact that $x' = \dfrac{1}{y'}$, we obtain that

$$G_z(y, x') = F(y, y') - y' F_z(y, y'). \tag{38}$$

In view of (36) and (38), we see that (35) follows from (36) and (38) by setting $C = -c_1$. □

6. Let V denote a normed, linear space, and V_o a nontrivial subspace of V, and Assume that $J : V \to \mathbb{R}$ and $J_1 : V \to \mathbb{R}$ are Gâteaux differentiable in V along any direction $v \in V_o$.

   (a) Show that $dJ(u; cv) = c\, dJ(u; v)$ for any $c \in \mathbb{R}$.

   ***Solution**:* Assume that J is Gâteaux differentiable at $u \in V$ in the direction $v \in V_o$. Then, $\lim\limits_{t\to 0} \dfrac{J(u + tv) - J(u)}{t}$ exists and

$$\lim_{t\to 0} \frac{J(u + tv) - J(u)}{t} = dJ(u; v). \tag{39}$$

Next, suppose that $c \neq 0$ and consider

$$\frac{J(u + t(cv)) - J(u)}{t} = c \cdot \frac{J(u + (ct)v)) - J(u)}{ct}, \quad \text{for } t \neq 0. \tag{40}$$

Put $r = ct$; then, since we are assuming that $c \neq 0$,

$$t \to 0 \text{ if and only if } r \to 0.$$

We then get from (40) that

$$\lim_{t\to 0} \frac{J(u + t(cv)) - J(u)}{t} = c \cdot \lim_{r\to 0} \frac{J(u + rv)) - J(u)}{r};$$

so that, in view of (39),

$$dJ(u; cv) = c \cdot dJ(u; v). \tag{41}$$

Observe that (41) also holds true for $c = 0$; this follows from the fact that $dJ(u; 0) = 0$, which can be seen to be true from the definition of the Gâteaux derivative in (39). □

(b) Show that $J + J_1$ is Gâteaux differentiable at any $u \in V$ in the direction of $v \in V_o$, and

$$d(J + J_1)(u; v) = dJ(u; v) + dJ_1(u; v). \tag{42}$$

***Solution**:* Let $u \in V$, $v \in V_o$ and assume that J and J_1 are Gâteaux differentiable at u in the direction of v. Then, the maps

$$t \mapsto J(u + tv), \quad \text{for } t \in \mathbb{R},$$

and

$$t \mapsto J_1(u + tv), \quad \text{for } t \in \mathbb{R},$$

are differentiable at $t = 0$ and

$$dJ(u; v) = \frac{d}{dt}[J(u + tv)]\Big|_{t=0} \tag{43}$$

and

$$dJ_1(u; v) = \frac{d}{dt}[J_1(u + tv)]\Big|_{t=0}. \tag{44}$$

Now, by the definition of the sum of the functionals J and J_1,

$$(J + J_1)(u + tv) = J(u + tv) + J_1(u + tv), \quad \text{for } t \in \mathbb{R},$$

so that, the map

$$t \mapsto (J + J_1)(u + tv), \quad \text{for } t \in \mathbb{R}, \tag{45}$$

is the sum of two differentiable functions of t; hence, the map in (45) is differentiable at $t = 0$ and

$$\frac{d}{dt}[(J + J_1)(u + tv)]\Big|_{t=0} = \frac{d}{dt}[J(u + tv)]\Big|_{t=0} + \frac{d}{dt}[J_1(u + tv)]\Big|_{t=0},$$

from which (42), by virtue of (43) and (44). □

7. Let V denote a normed, linear space, and V_o a nontrivial subspace of V, and $\mathcal{A}$ a nonempty subset of V. Assume that $J_1 \colon V \to \mathbb{R}$ and $J_2 \colon V \to \mathbb{R}$ are Gâteaux differentiable in V along any direction $v \in V_o$.

(a) Show that if J_1 and J_2 are convex in $\mathcal{A}$, then so are $c^2 J_1$ and $J_1 + J_2$, for any $c \in \mathbb{R}$.

***Solution**:* Assume that $J_1 \colon V \to \mathbb{R}$ and $J_2 \colon V \to \mathbb{R}$ are convex functionals in $\mathcal{A}$. Then,

$$J_1(u+v) \geqslant J_1(u) + dJ_1(u;v), \quad \text{for } u \in \mathcal{A},\ v \in V_o \text{ with } u+v \in \mathcal{A}, \tag{46}$$

and

$$J_2(u+v) \geqslant J_2(u) + dJ_2(u;v), \quad \text{for } u \in \mathcal{A},\ v \in V_o \text{ with } u+v \in \mathcal{A}. \tag{47}$$

Multiply on both sides of (46) by c^2 to get

$$c^2 J_1(u+v) \geqslant c^2 J_1(u) + c^2 dJ_1(u;v), \quad \text{for } u \in \mathcal{A},\ v \in V_o \text{ with } u+v \in \mathcal{A},$$

since $c^2 \geqslant 0$. Then, since

$$d(c^2 J(u;v)) = c^2 dJ(u;v),$$

we get that

$$c^2 J_1(u+v) \geqslant c^2 J_1(u) + d(vJ_1(u;v)), \quad \text{for } u \in \mathcal{A},\ v \in V_o \text{ with } u+v \in \mathcal{A},$$

which shows that $c^2 J_1$ is convex in $\mathcal{A}$.

Next, use the definition of the sum of functionals to compute, for $u \in \mathcal{A}$ and $v \in V_o$ with $u+v \in \mathcal{A}$,

$$(J_1 + J_2)(u+v) = J_1(u+v) + J_2(u+v);$$

so that, in view of (46) and (47),

$$\begin{aligned} (J_1 + J_2)(u+v) &\geqslant J_1(u) + dJ_1(u;v) + J_2(u) + dJ_2(u;v) \\ &= J_1(u) + J_2(u) + dJ_1(u;v) + dJ_2(u;v) \\ &= (J_1 + J_2)(u) + d(J_1 + J_2)(u;v), \end{aligned}$$

where we have used the result (42) in part (b) Problem 6. Thus,

$$(J_1 + J_2)(u+v) \geqslant (J_1 + J_2)(u) + d(J_1 + J_2)(u;v), \tag{48}$$

for $u \in \mathcal{A}$, $v \in V_o$ with $u+v \in \mathcal{A}$. Hence, $J_1 + J_2$ is convex in $\mathcal{A}$. $\square$

(b) Show that if J_1 is convex in $\mathcal{A}$ and J_2 is strictly convex in $\mathcal{A}$, then $J_1 + J_2$ is strictly convex in $\mathcal{A}$.

***Solution**:* Assume $J_1 \colon V \to \mathbb{R}$ is convex in $\mathcal{A}$ and $J_2 \colon V \to \mathbb{R}$ is strictly convex in $\mathcal{A}$. Then, the inequalities in (46) and (47) hold true, and equality in (47) holds true if and only if $v = 0$.

We have already seen in part (a) that $J_1 + J_2$ is convex in $\mathcal{A}$, To show that $J_1 + J_2$ is also strictly convex in $\mathcal{A}$, suppose that equality in (48) holds true; so that,

$$(J_1 + J_2)(u + v) = (J_1 + J_2)(u) + d(J_1 + J_2)(u; v),$$

for $u \in \mathcal{A}$, and $v \in V_o$ with $u + v \in \mathcal{A}$. Then, using (42) and the definition of the sum of two functionals,

$$J_1(u + v) + J_2(u + v) = J_1(u) + J_2(u) + dJ_1(u; v) + dJ_2(u; v),$$

which we can rewrite as

$$J_2(u + v) - J_2(u) - dJ_2(u; v) = J_1(u) + dJ_1(u; v) - J_1(u + v). \tag{49}$$

It follows from the inequalities in (47) and (46) that the left–hand side of (49) is greater that or equal to 0, while the right–hand side is less than or equal to 0. Consequently,

$$J_2(u + v) - J_2(u) - dJ_2(u; v) = 0.$$

Therefore, since J_2 is strictly convex in $\mathcal{A}$, it follows that $v = 0$. We have thus shown that, if equality in (48) holds true, then $v = 0$. Conversely, we can see that if $v = 0$ then equality in (48) holds true. Hence, $J_1 + J_2$ is strictly convex in $\mathcal{A}$, if J_2 is strictly convex in $\mathcal{A}$. □

8. Let $J \colon \mathcal{A} \to \mathbb{R}$ be defined by

$$J(y) = \int_0^{1/2} [y(x) + \sqrt{1 + (y'(x))^2}]\, dx, \quad \text{for all } y \in \mathcal{A}, \tag{50}$$

where

$$\mathcal{A} = \{y \in C^1([0, 1/2], \mathbb{R}) \mid y(0) = -1 \text{ and } y(1/2) = -\sqrt{3}/2\}. \tag{51}$$

Verify that J is strictly convex and find, if possible, the unique minimizing function for J in $\mathcal{A}$.

***Solution**:* Write the functional J in (50) as the sum of two functionals $J_1\colon V \to \mathbb{R}$ and $J_2\colon V \to \mathbb{R}$, where $V = C^1([0,1/2],\mathbb{R})$ and

$$J_1(y) = \int_0^{1/2} y(x)\ dx, \qquad \text{for } y \in V, \tag{52}$$

and

$$J_2(y) = \int_0^{1/2} \sqrt{1+(y'(x))^2}\ dx, \qquad \text{for } y \in V. \tag{53}$$

In view of (50), (52) and (53), we see that

$$J(y) = J_1(y) + J_2(y), \qquad \text{for all } y \in V.$$

Observe that J_1 is a linear functional; hence, J_1 is convex (but not strictly convex). On the other hand, J_2 is strictly convex; this is a consequence of the result in part (a) of Problem 4 in this set of problems, with $f(z) = \sqrt{1+z^2}$ for all $z \in \mathbb{R}$. Hence, by the result in part (b) in Problem 7, the functional J defined in (50) is strictly convex in $\mathcal{A}$.

To find the unique minimizer in $\mathcal{A}$ of the functional J defined in (50), we first solve the Euler–Lagrange equation associated with J. In this case, the function $F(x,y,z)$ corresponding to J is

$$F(x,y,z) = y + \sqrt{1+z^2}, \qquad \text{for } x \in [0,1/2], y \in \mathbb{R}, \text{ and } z \in \mathbb{R}.$$

The partial derivatives of F with respect to y and z, respectively, are

$$F_y(x,y,z) = 1 \quad\text{and}\quad F_z(x,y,z) = \frac{z}{\sqrt{1+z^2}}, \quad \text{for } (x,y,z) \in [0,1/2]\times\mathbb{R}\times\mathbb{R};$$

so that, the Euler–Lagrange equation associated with the functional J given in (50) is

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 1. \tag{54}$$

Integrating the equation in (54) yields

$$\frac{y'}{\sqrt{1+(y')^2}} = x + c_1, \qquad \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \tag{55}$$

and for some constant of integration c_1. Solving the equation in (55) for y', we obtain

$$y' = \frac{x+c_1}{\sqrt{1-(x+c_1)^2}}. \tag{56}$$

The differential equation in (56) can be integrated to yield

$$y(x) = -\sqrt{1-(x+c_1)^2} + c_2, \qquad \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \tag{57}$$

and another constant of integration c_2.

Since we are looking for a minimizer of J in $\mathcal{A}$, where $\mathcal{A}$ is given in (51), the function in (57) must satisfy the boundary conditions

$$y(0) = -1 \quad \text{and} \quad y(1/2) = -\frac{\sqrt{3}}{4}. \tag{58}$$

Thus, we must have that $c_1 = c_2 = 0$ in (57) for y to satisfy the boundary conditions in (58). Consequently,

$$y(x) = -\sqrt{1-x^2}, \qquad \text{for } 0 \leqslant x \leqslant \frac{1}{2}. \tag{59}$$

By the Convex Minimization Theorem proved in class and in the lecture notes, the function $y \in \mathcal{A}$ given in (59) is the unique minimizer of the functional J defined in (50) over $\mathcal{A}$ given in (51). □

9. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by

$$J(y) = \int_1^2 [y(x) + xy'(x)]\, dx, \qquad \text{for all } y \in \mathcal{A}, \tag{60}$$

where

$$\mathcal{A} = \{y \in C^1[1,2] \mid y(1) = 1 \text{ and } y(2) = 2\}. \tag{61}$$

Verify that J is convex, but not strictly convex in $\mathcal{A}$. Can you find more than one function which minimizes J in $\mathcal{A}$?

***Solution**:* Observe that the functional J defined in (60) and (61) is a linear functional in $V = C^1([1,2],\mathbb{R})$. Hence, J is is convex, but not strictly convex in $\mathcal{A}$.

Observe also that, for any $y \in \mathcal{A}$,

$$J(y) = \int_1^2 \frac{d}{dx}[xy(x)]\, dx,$$

where we have used the Product Rule; so that, by the Fundamental Theorem of Calculus,

$$J(y) = xy(x)\Big|_1^2 = 2y(2) - y(1).$$

Hence, using the definition of $\mathcal{A}$ in (61),

$$J(y) = 2(2) - 1 = 3, \qquad \text{for all } y \in \mathcal{A};;$$

so that J is constant in $\mathcal{A}$. Hence, every function in $\mathcal{A}$ is a minimizer of J in $\mathcal{A}$. $\square$