

## Assignment #2

Due on Friday, September 14, 2018

**Read** Section 1.2 on *The Vector Space*  $\mathbb{R}^n$  in Baxandall and Liebek's text (pp. 2–9).

**Read** Sections 2.3 on *The Dot Product and Euclidean Norm* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

**Read** Sections 2.4 on *Orthogonality and Projections* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

**Do** the following problems

1. The dot product, or inner product, of two vectors in  $\mathbb{R}^n$  is symmetric, bi-linear and positive definite; that is, for vectors  $v$ ,  $v_1$ ,  $v_2$  and  $w$  in  $\mathbb{R}^n$ ,
  - (i)  $v \cdot w = w \cdot v$
  - (ii)  $(c_1v_1 + c_2v_2) \cdot w = c_1v_1 \cdot w + c_2v_2 \cdot w$ , and
  - (iii)  $v \cdot v \geq 0$  for all  $v \in \mathbb{R}^n$  and  $v \cdot v = 0$  if and only if  $v$  is the zero vector.

Use these properties of the the inner product in  $\mathbb{R}^n$  to derive the following properties of the norm  $\|\cdot\|$  in  $\mathbb{R}^n$ , where

$$\|v\| = \sqrt{v \cdot v} \quad \text{for all vectors } v \in \mathbb{R}^n.$$

- (a)  $\|v\| \geq 0$  for all  $v \in \mathbb{R}^n$  and  $\|v\| = 0$  if and only if  $v = \vec{0}$ .
  - (b) For a scalar  $c$ ,  $\|cv\| = |c|\|v\|$ .
2. **The Cauchy-Schwarz inequality.** For any vectors  $v$  and  $w$  in  $\mathbb{R}^n$ ,

$$|v \cdot w| \leq \|v\|\|w\|.$$

Use this inequality to derive the triangle inequality: For any vectors  $v$  and  $w$  in  $\mathbb{R}^n$ ,

$$\|v + w\| \leq \|v\| + \|w\|.$$

(*Suggestion:* Start with the expression  $\|v + w\|^2$  and use the properties of the inner product to simplify it.)

3. Given two non-zero vectors  $v$  and  $w$  in  $\mathbb{R}^n$ , the cosine of the angle,  $\theta$ , between the vectors can be defined by

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|}.$$

Use the Cauchy-Schwarz inequality to justify why this definition makes sense.

4. Two vectors  $v$  and  $w$  in  $\mathbb{R}^n$  are said to be *orthogonal* or perpendicular, if and only if  $v \cdot w = 0$ .

Show that if  $v$  and  $w$  are orthogonal, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$

Give a geometric interpretation of this result in two-dimensional Euclidean space.

5. A vector  $u$  in  $\mathbb{R}^n$  is said to be a unit vector if and only if  $\|u\| = 1$ . Let  $u$  be a unit vector in  $\mathbb{R}^n$  and  $v$  be any vector in  $\mathbb{R}^n$ .

- (a) Give the parametric equation of the line through origin in the direction of  $u$ .
- (b) Let  $f(t) = \|v - tu\|^2$  for all  $t \in \mathbb{R}$ . Explain why this function gives the square of the distance from the point at  $v$  to a point on the line through the origin in the direction of  $u$ .
- (c) Show that  $f(t)$  is minimized when  $t = v \cdot u$ .