

Assignment #5

Due on Friday, September 28, 2018

Read Section 2.1 on *Vector-Valued Functions of \mathbb{R}* in Baxandall and Liebek's text (pp. 26–29).

Read Section 4.1 on *Vector-Valued Functions of \mathbb{R}^m* in Baxandall and Liebek's text (pp. 182–184).

Read Section 4.2 on *Continuity and Limits* in Baxandall and Liebek's text (pp. 185–188).

Read Section 3.1 on *Types of Functions in Euclidean Space* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Read Section 3.2 on *Open Subsets of Euclidean Space* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Do the following problems

1. Let U_1 and U_2 denote subsets in \mathbb{R}^n .
 - (a) Show that if U_1 and U_2 are open subsets of \mathbb{R}^n , then their intersection

$$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

is also open.

- (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\}$ is not an open subset of \mathbb{R}^2 .

2. In Problem 4 of Assignment #3 you proved that every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}$ must be of the form

$$T(v) = w \cdot v \quad \text{for every } v \in \mathbb{R}^n,$$

where w is some vector in \mathbb{R}^n . Use this fact, together with the Cauchy–Schwarz inequality, to prove that T is continuous at every point in \mathbb{R}^n .

3. A subset, U , of \mathbb{R}^n is said to be **convex** if given any two points x and y in U , the straight line segment connecting them is entirely contained in U ; in symbols,

$$\{x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leq 1\} \subseteq U$$

- (a) Prove that the ball $B_r(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is a convex subset of \mathbb{R}^n .
(b) Prove that the “punctured unit disc” in \mathbb{R}^2 ,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1 \right\},$$

is not a convex set.

4. Let x and y denote real numbers.

- (a) Starting with the self-evident inequality: $(|x| - |y|)^2 \geq 0$, derive the inequality

$$|xy| \leq \frac{1}{2}(x^2 + y^2).$$

- (b) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

Use the inequality derived in the previous part to prove that f is continuous at the origin.

5. Let

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

Define $f(0, 0)$ so that $f(x, y)$ is continuous at $(0, 0)$. Justify your answer.