Solutions to Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

***Solution**:* The point $P_o(3,0,0)$ is in the plane. Let
$$w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$$

The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, d, from P to the plane, we compute the norm of the orthogonal projection of w onto n; that is,
$$d = \|\mathrm{P}_{\widehat{n}}(w)\|,$$
where
$$\widehat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$
a unit vector in the direction of n, and
$$\mathrm{P}_{\widehat{n}}(w) = (w \cdot \widehat{n})\widehat{n}.$$

It then follows that
$$d = |w \cdot \widehat{n}|,$$
where $w \cdot \widehat{n} = \dfrac{1}{\sqrt{26}}(4+21) = \dfrac{25}{\sqrt{26}}$. Hence, $d = \dfrac{25\sqrt{26}}{26} \approx 4.9.$ $\square$

2. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x = -1 + 4t, \\ y = -7t, \\ z = 2 - t. \end{cases}$$

Solution: The point $P_o(-1,0,2)$ is on the line. The vector

$$v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix}$$

gives the direction of the line. Put

$$w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}.$$

The vectors v and w determine a parallelogram whose area is the norm of v times the shortest distance, d, from P to the line determined by v at P_o. We then have that

$$\text{area}(P(v,w)) = \|v\|d,$$

from which we get that

$$d = \frac{\text{area}(P(v,w))}{\|v\|}.$$

On the other hand,

$$\text{area}(P(v,w)) = \|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\widehat{i} + 31\widehat{j} + 35\widehat{k}.$$

Thus, $\|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155}$ and therefore

$$d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.$$

□

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1,1,0)$, $(2,0,1)$ and $(0,3,1)$

***Solution**:* Label the points $P_o(1,1,0)$, $P_1(2,0,1)$ and $P_2(0,3,1)$ and define the vectors

$$v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The area of the triangle determined by the points P_o, P_1 and P_2 is then half of the area of the parallelogram determined by the vectors v and w. Thus,

$$\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\widehat{i} - 2\widehat{j} + \widehat{k}.$$

Consequently, $\text{area}(\triangle P_oP_1P_2) = \dfrac{1}{2}\sqrt{9+4+1} = \dfrac{\sqrt{14}}{2} \approx 1.87.$ □

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

***Solution**:* The area of the parallelogram determined by v and $w+\lambda v$ is

$$\text{area}(P(v, w+\lambda v)) = \|v \times (w + \lambda v)\|,$$

where

$$v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w.$$

Consequently, $\text{area}(P(v, w+\lambda v)) = \|v \times w\| = \text{area}(P(v, w)).$ □

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Solution: $P_{\widehat{u}}(v) = (v \cdot \widehat{u})\widehat{u}$ for all $v \in \mathbb{R}^n$. Consequently, for any $w, v \in \mathbb{R}^n$,

$$\begin{aligned} P_{\widehat{u}}(w) - P_{\widehat{u}}(v) &= (w \cdot \widehat{u})\widehat{u} - (v \cdot \widehat{u})\widehat{u} \\ &= (w \cdot \widehat{u} - v \cdot \widehat{u})\widehat{u} \\ &= [(w - v) \cdot \widehat{u}]\widehat{u}. \end{aligned}$$

It then follows that

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = |(w - v) \cdot \widehat{u}|,$$

since $\|\widehat{u}\| = 1$. Hence, by the Cauchy–Schwarz inequality,

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| \leqslant \|w - v\|.$$

Applying the Squeeze Theorem we then get that

$$\lim_{\|w-v\| \to 0} \|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = 0,$$

which shows that $P_{\widehat{u}}$ is continuous at every $v \in V$. □

6. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that f is continuous at $(0, 0)$.

Solution: For $(x, y) \neq (0, 0)$

$$\begin{aligned} |f(x, y)| &= \frac{x^2 |y|}{x^2 + y^2} \\ &\leqslant |y|, \end{aligned}$$

since $x^2 \leqslant x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$. We then have that, for $(x, y) \neq (0, 0)$,

$$|f(x, y)| \leqslant \sqrt{x^2 + y^2},$$

which implies that

$$0 \leqslant |f(x, y) - f(0, 0)| \leqslant \|(x, y) - (0, 0)\|,$$

for $(x, y) \neq (0, 0)$. Thus, by the Squeeze Theorem,

$$\lim_{\|(x,y)-(0,0)\| \to 0} |f(x, y) - f(0, 0)| = 0,$$

which shows that f is continuous at $(0, 0)$. □

7. Show that

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$.

Solution: Let $\sigma_1(t) = (t, t)$ for all $t \in \mathbb{R}$ and observe that

$$\lim_{t \to 0} \sigma_1(t) = (0, 0)$$

and

$$f(\sigma(t)) = 0, \quad \text{for all } t \neq 0.$$

It then follows that

$$\lim_{t \to 0} f(\sigma_1(t)) = 0.$$

Thus, if f were continuous at $(0, 0)$, we would have that

$$f(0, 0) = 0. \tag{1}$$

On the other hand, if we let $\sigma_2(t) = (t, 0)$, we would have that

$$\lim_{t \to 0} \sigma_2(t) = (0, 0)$$

and

$$f(\sigma(t)) = 1, \quad \text{for all } t \neq 0.$$

Thus, if f were continuous at $(0, 0)$, we would have that

$$f(0, 0) = 1,$$

which is in contradiction with (1). This contradiction shows that f is not continuous at $(0, 0)$. □

8. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x \ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y \neq 0; \\ \\ L & \text{otherwise} \ , \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

***Solution**:* Observe that, for $y \neq 0$,

$$\begin{aligned} |f(x,y)| &= \left| x \ \sin\left(\frac{1}{y}\right)\right| \\ &= |x| \ \left|\sin\left(\frac{1}{y}\right)\right| \\ &\leqslant |x| \\ &\leqslant \sqrt{x^2+y^2}. \end{aligned}$$

It then follows that, for $y \neq 0$,

$$0 \leqslant |f(x,y)| \leqslant \|(x,y)\|.$$

Consequently, by the Squeeze Theorem,

$$\lim_{\|(x,y)\|\to 0} |f(x,y)| = 0.$$

This suggests that we define $L = 0$. If this is the case,

$$\lim_{\|(x,y)\|\to 0} |f(x,y) - f(0,0)| = 0,$$

which shows that f is continuous at $(0,0)$ if $L = 0$.

Next, assume now that $L = 0$ in the definition of f. Then, for any $a \neq 0$, f fails for be continuous at $(a,0)$. To see why this is case, note that for any $y \neq 0$

$$f(a,y) = a \ \sin\left(\frac{1}{y}\right)$$

and the limit of $\sin\left(\dfrac{1}{y}\right)$ as $y \to 0$ does not exist. $\square$

9. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

***Solution**:* Let u and w be any vector in $\mathbb{R}^n$ and consider

$$\begin{aligned} f(u+w) &= \frac{1}{2}\|u+w\|^2 \\ &= \frac{1}{2}(u+w)\cdot(u+w) \\ &= \frac{1}{2}u\cdot u + u\cdot w + \frac{1}{2}w\cdot w \\ &= \frac{1}{2}\|u\|^2 + u\cdot w + \frac{1}{2}\|w\|^2. \end{aligned}$$

Thus,

$$f(u+w) - f(u) - u\cdot w = \frac{1}{2}\|w\|^2.$$

Consequently,

$$\frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = \frac{1}{2}\|w\|,$$

for $w \in \mathbb{R}^n$ with $\|w\| \neq 0$, from which we get that

$$\lim_{\|w\|\to 0} \frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = 0,$$

and therefore f is differentiable at u with derivative map $Df(u)$ given by

$$Df(u)w = u\cdot w \quad \text{for all } w \in \mathbb{R}^n.$$

Hence, $\nabla f(u) = u$ for all $u \in \mathbb{R}^n$. □

10. Let $g\colon\ [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

    (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

Solution: Take the partial derivative of $r^2 = x^2 + y^2$ on both sides with respect to x to obtain

$$\frac{\partial(r^2)}{\partial x} = 2x.$$

Applying the chain rule on the left–hand side we get

$$2r\frac{\partial r}{\partial x} = 2x,$$

which leads to

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$. □

(b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

Solution: Take the partial derivative of $f(x, y) = g(r)$ on both sides with respect to x and apply the Chain Rule to obtain

$$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}.$$

Similarly, $\dfrac{\partial f}{\partial y} = g'(r)\dfrac{y}{r}$.

It then follows that

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\widehat{i} + \frac{\partial f}{\partial y}\widehat{j} \\ &= g'(r)\frac{x}{r}\widehat{i} + g'(r)\frac{y}{r}\widehat{j} \\ &= \frac{g'(r)}{r}(x\widehat{i} + y\widehat{j}) \\ &= \frac{g'(r)}{r}\mathbf{r}. \end{aligned}$$

□

11. Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset U of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0}\frac{f(v + t\widehat{u}) - f(v)}{t}$$

exists, we call it the *directional derivative of f at v in the direction of the unit vector $\widehat{u}$*. We denote it by $D_{\widehat{u}}f(v)$.

(a) Show that if f is differentiable at $v \in U$, then, for any unit vector $\widehat{u}$ in $\mathbb{R}^n$, the directional derivative of f in the direction of $\widehat{u}$ at v exists, and

$$D_{\widehat{u}} f(v) = \nabla f(v) \cdot \widehat{u},$$

where $\nabla f(v)$ is the gradient of f at v.

Proof: Suppose that f is differentiable at $v \in U$. Then,

$$f(v+w) = f(v) + Df(v)w + E(w),$$

where

$$Df(v)w = \nabla f(v) \cdot w,$$

and

$$\lim_{\|w\| \to 0} \frac{|E(w)|}{\|w\|} = 0.$$

Thus, for any $t \in \mathbb{R}$,

$$f(v + t\widehat{u}) = f(v) + t\nabla f(v) \cdot \widehat{u} + E(t\widehat{u}),$$

where

$$\lim_{|t| \to 0} \frac{|E(t\widehat{u})|}{|t|} = 0,$$

since $\|t\widehat{u}\| = |t|\|\widehat{u}\| = |t|$.
We then have that, for $t \neq 0$,

$$\frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} = \frac{E(t\widehat{u})}{t},$$

and consequently

$$\left| \frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} \right| = \frac{|E(t\widehat{u})|}{|t|},$$

from which we get that

$$\lim_{t \to 0} \left| \frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} \right| = 0.$$

□

(b) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\widehat{u}} f(v) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

Proof: Suppose, by way of contradiction, that $\nabla f(v) \neq \mathbf{0}$, and put

$$\widehat{u} = \frac{1}{\|\nabla f(v)\|}\nabla f(v).$$

Then, $\widehat{u}$ is a unit vector, and therefore, by the assumption,

$$D_{\widehat{u}} f(v) = 0,$$

or

$$\nabla f(v) \cdot \widehat{u} = 0.$$

But this implies that

$$\nabla f(v) \cdot \frac{1}{\|\nabla f(v)\|}\nabla f(v) = 0,$$

where

$$\begin{aligned} \nabla f(v) \cdot \frac{1}{\|\nabla f(v)\|}\nabla f(v) &= \frac{1}{\|\nabla f(v)\|}\nabla f(v) \cdot \nabla f(v) \\ &= \frac{1}{\|\nabla f(v)\|}\|\nabla f(v)\|^2 \\ &= \|\nabla f(v)\|. \end{aligned}$$

It then follows that $\|\nabla f(v)\| = 0$, which contradicts the assumption that $\nabla f(v) \neq \mathbf{0}$. Therefore, $\nabla f(v)$ must be the zero vector. □

(c) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}} f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(v)$.

Proof. If f is differentiable at x, then $D_{\widehat{u}} f(x) = \nabla f(x) \cdot \widehat{u}$, as was shown in part (a). Thus, by the Cauchy–Schwarz inequality,

$$|D_{\widehat{u}} f(x)| \leqslant \|\nabla f(x)\|\|\widehat{u}\| = \|\nabla f(x)\|,$$

since $\widehat{u}$ is a unit vector. Hence,

$$-\|\nabla f(x)\| \leqslant D_{\widehat{u}} f(x) \leqslant \|\nabla f(x)\|$$

for any unit vector $\widehat{u}$, and so the largest value that $D_{\widehat{u}} f(x)$ can have is $\|\nabla f(x)\|$.

If $\nabla f(x) \neq \mathbf{0}$, then $\widehat{u} = \dfrac{1}{\|\nabla f(x)\|}\nabla f(x)$ is a unit vector, and

$$\begin{aligned} D_{\widehat{u}} f(x) &= \nabla f(x) \cdot \widehat{u} \\ &= \nabla f(x) \cdot \frac{1}{\|\nabla f(x)\|}\nabla f(x) \\ &= \frac{1}{\|\nabla f(x)\|}\nabla f(x) \cdot \nabla f(x) \\ &= \frac{1}{\|\nabla f(x)\|}\|\nabla f(x)\|^2 \\ &= \|\nabla f(x)\|. \end{aligned}$$

Thus, $D_{\widehat{u}} f(x)$ attains its largest value when $\widehat{u}$ is in the direction of $\nabla f(x)$.

□