Review Problems for Exam 2

1. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y-x)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function g is well defined.

   (b) Show that g is differentiable on $(0,1)$ and that

$$g'(t) = \nabla f(x + t(y-x)) \cdot (y-x) \quad \text{for} \quad 0 < t < 1.$$

   (c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}} f(z) \|y-x\|,$$

   where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y-x\|}(y-x)$.

   (d) Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

2. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

   Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

3. Let C denote the boundary of the oriented triangle, $T = [(0,0)(1,0)(1,2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\displaystyle\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x$, by applying the Fundamental Theorem of Calculus.

4. Let $F(x,y) = 2x\ \widehat{i} - y\ \widehat{j}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

5. Evaluate the line integral $\displaystyle\int_{\partial R}(x^4+y)\ \mathrm{d}x+(2x-y^4)\ \mathrm{d}y$, where R is the rectangular region

$$R=\{(x,y)\in\mathbb{R}^2\mid -1\leqslant x\leqslant 3,\ -2\leqslant y\leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

6. Integrate the function given by $f(x,y)=xy^2$ over the region, R, defined by:

$$R=\{(x,y)\in\mathbb{R}^2\mid x\geqslant 0, 0\leqslant y\leqslant 4-x^2\}.$$

7. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

for $a>0$ and $b>0$.

   (a) Evaluate the line integral $\displaystyle\oint_{\partial R} x\ \mathrm{d}y-y\ \mathrm{d}x$, where ∂R is the ellipse in (1) traversed in the positive sense.

   (b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

8. Evaluate the double integral $\displaystyle\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 1.

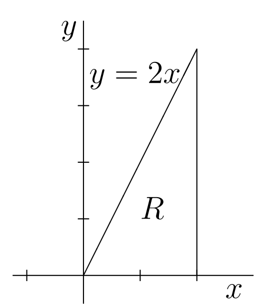

Figure 1: Sketch of Region R in Problem 8