Solutions to Review Problems for Exam 2

1. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

(a) Explain why the function g is well defined.

Answer: Since U is convex, for any $x, y \in U$, $x + t(y - x) \in U$ for all $t \in [0, 1]$. Thus, $f(x + t(y - x))$ is defined for all $t \in [0, 1]$, because f is defined on U. □

(b) Show that g is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

Solution: It follows from the Chain Rule that the composition $g = f \circ \sigma\colon [0, 1] \to \mathbb{R}$, where $\sigma\colon [0, 1] \to \mathbb{R}^n$ is the path given by

$$\sigma(t) = x + t(y - x), \qquad \text{for all } t \in [0, 1],$$

is differentiable and

$$g'(t) = \nabla f(\sigma(t)) \cdot \sigma'(t), \qquad \text{for all } t \in (0, 1),$$

where

$$\sigma(t) = y - x, \qquad \text{for all } t.$$

Consequently, we get that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

□

(c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}} f(z) \|y - x\|, \tag{1}$$

where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$.

Solution: The mean value theorem implies that there exists $\tau \in (0, 1)$ such that

$$g(1) - g(0) = g'(\tau)(1 - 0),$$

so that

$$f(y) - f(x) = \nabla f(x + \tau(y - x)) \cdot (y - x). \tag{2}$$

Put $z = x + \tau(y - x)$ and $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$. We can then write (2) as

$$\begin{aligned} f(y) - f(x) &= \left(\nabla f(z) \cdot \frac{1}{\|y - x\|}(y - x)\right) \|y - x\| \\ &= (\nabla f(z) \cdot \widehat{u}) \|y - x\|, \end{aligned}$$

which yields (1). □

(d) Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f \colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

Solution: Fix $x_o \in U$. Then, for any $x \in U$, the formula in (1) yields

$$f(x) - f(x_o) = D_{\widehat{u}} f(z) \|x - x_o\|, \tag{3}$$

where $D_{\widehat{u}} f(z) = \nabla f(z) \cdot \widehat{u} = 0$ by the assumption. Hence, it follows from (3) that

$$f(x) = f(x_o), \quad \text{for all } x \in U;$$

in other words, f is constant in U. □

2. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f \colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma \colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

   Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

***Solution**:* If f has a local maximum or minimum at $\sigma(t_o)$, then $g'(t_o) = 0$, where, by the Chain rule,

$$g'(t) = \nabla f(\sigma(t)) \cdot \sigma'(t) \quad \text{for all} \quad t \in I.$$

It then follows that

$$\nabla f(\sigma(t_o)) \cdot \sigma'(t_o) = 0,$$

and, consequently, $\nabla f(\sigma(t_o)$ is perpendicular to the tangent to the path at $\sigma(t_o)$. □

3. Let C denote the boundary of the oriented triangle, $T = [(0,0)(1,0)(1,2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x$, by applying the Fundamental Theorem of Calculus.

***Solution**:* Apply the Fundamental Theorem of Calculus to the 1–form

$$\omega = -\frac{y^2}{2}\ \mathrm{d}x + \frac{x^2}{2}\ \ \mathrm{d}y$$

over the oriented triangle T; namely,

$$\int_{\partial T} \omega = \int_T d\omega,$$

where

$$d\omega = (x+y)\ dx \wedge dy.$$

Thus, since T is positively oriented, it follows that

$$\begin{aligned}
\int_{\partial T} \omega &= \iint_T (x+y)\ dxdy \\
&= \int_0^1 \int_0^{2x} (x+y)\ dydx \\
&= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{2x} dx \\
&= \int_0^1 4x^2 dx,
\end{aligned}$$

so that

$$\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x = \frac{4}{3}.$$

$\square$

4. Let $F(x,y) = 2x\ \widehat{i} - y\ \widehat{j}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

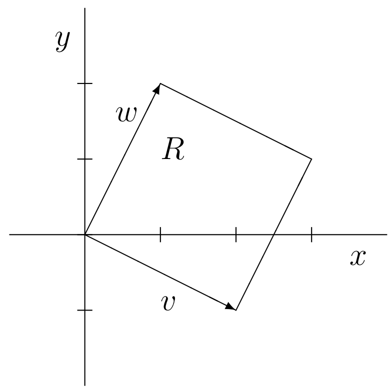

Figure 1: Sketch of Region R in Problem 4

***Solution**:* Apply the Fundamental Theorem of Calculus,

$$\oint_{\partial R} F \cdot \widehat{n}\ ds = \int_R d\omega,$$

where

$$\omega = P\ dy - Q\ dx = 2x\ dy - (-y)\ dx = y\ dx + 2x\ dy,$$

so that

$$d\omega = dy \wedge dx + 2dx \wedge dy = dx \wedge dy,$$

we obtain that

$$\begin{aligned} \oint_{\partial R} F \cdot\ d\mathbf{n} &= \int_R dx \wedge dy \\ &= \iint_R dxdy \\ &= \text{area}(R). \end{aligned}$$

To find the area of the region R, shown in Figure 1, observe that R is a parallelogram determined by the vectors $v = 2\,\widehat{\imath} - \widehat{\jmath}$ and $w = \widehat{\imath} + 2\,\widehat{\jmath}$. Thus,

$$\text{area}(R) = \|v \times w\| = 5.$$

It the follows that

$$\oint_{\partial R} F \cdot n \ \ \mathrm{d}s = \iint_R \mathrm{d}x\ \mathrm{d}y = 5.$$

□

5. Evaluate the line integral $\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

***Solution**:* Apply the Fundamental Theorem of Calculus to get

$$\begin{aligned}
\int_{\partial R} (x^4 + y)\ dx + (2x - y^4)\ dy &= \int_R d(x^4 + y) \wedge dx + d(2x - y^4) \wedge dy \\
&= \int_R dy \wedge dx + 2dx \wedge dy \\
&= \int_R dx \wedge dy \\
&= \text{area}(R) \\
&= 12.
\end{aligned}$$

□

6. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

***Solution**:* The region, R, is sketched in Figure 2. We evaluate the

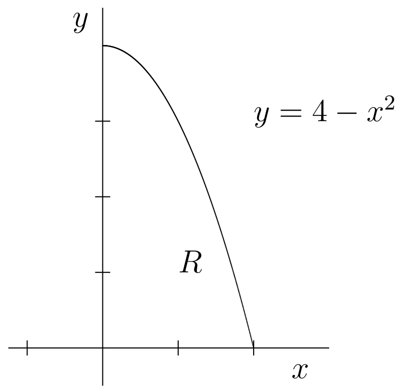

Figure 2: Sketch of Region R in Problem 8

double integral, $\displaystyle\iint_R xy^2 \;\mathrm{d}x\;\mathrm{d}y$, as an iterated integral

$$
\begin{aligned}
\iint_R xy^2 \;\mathrm{d}x\;\mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \;\mathrm{d}y\;\mathrm{d}x \\
&= \int_0^2 \int_0^{4-x^2} xy^2 \;\mathrm{d}y\;\mathrm{d}x \\
&= \int_0^2 \frac{xy^3}{3}\bigg|_0^{4-x^2} \;\mathrm{d}x \\
&= \frac{1}{3}\int_0^2 x(4-x^2)^3 \;\mathrm{d}x.
\end{aligned}
$$

To evaluate the last integral, make the change of variables: $u = 4 - x^2$. We then have that $\mathrm{d}u = -2x\;\mathrm{d}x$ and

$$
\begin{aligned}
\iint_R xy^2 \;\mathrm{d}x\;\mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \;\mathrm{d}y\;\mathrm{d}x \\
&= -\frac{1}{6}\int_4^0 u^3 \;\mathrm{d}u \\
&= \frac{1}{6}\int_0^4 u^3 \;\mathrm{d}u.
\end{aligned}
$$

Thus,

$$\iint_R xy^2 \; \mathrm{d}x \; \mathrm{d}y \;\; = \;\; \frac{4^4}{24} = \frac{32}{3}.$$

□

7. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x \; \mathrm{d}y - y \; \mathrm{d}x$, where ∂R is the ellipse in (4) traversed in the positive sense.

Solution: A sketch of the ellipse is shown in Figure 3 for the case $a < b$.

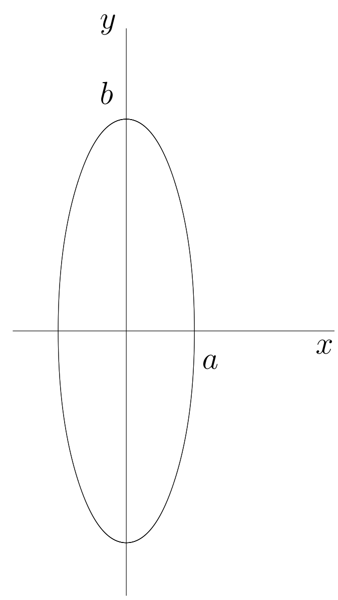

Figure 3: Sketch of ellipse

A parametrization of the ellipse is given by

$$x = a\cos t, \quad y = b\sin t, \quad \text{for} \;\; 0 \leqslant t \leqslant 2\pi.$$

We then have that $\mathrm{d}x = -a\sin t\,\mathrm{d}t$ and $\mathrm{d}y = b\cos t\,\mathrm{d}t$. Therefore

$$\begin{aligned}\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x &= \int_0^{2\pi} [a\cos t\cdot b\cos t - b\sin t\cdot(-a\cos t)]\ \mathrm{d}t \\ &= \int_0^{2\pi} [ab\cos^2 t + ab\sin^2 t]\ \mathrm{d}t \\ &= ab\int_0^{2\pi} (\cos^2 t + ab\sin^2 t)\ \mathrm{d}t \\ &= ab\int_0^{2\pi}\ \mathrm{d}t \\ &= 2\pi ab.\end{aligned}$$

□

(b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (4).

Solution: Let $F(x, y) = x\ \widehat{i} + y\ \widehat{j}$. Then,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \oint_{\partial R} F\cdot n\ \ \mathrm{d}s.$$

Thus, by Green's Theorem in divergence form,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\mathrm{div}F(x, y) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

Consequently,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = 2\iint_R\ \mathrm{d}x\ \mathrm{d}y = 2\ \mathrm{area}(R).$$

It then follows that

$$\mathrm{area}(R) = \frac{1}{2}\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x.$$

Thus,

$$\mathrm{area}(R) = \pi ab,$$

by the result in part (a). □

8. Evaluate the double integral $\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 4.

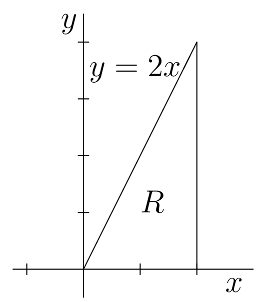

Figure 4: Sketch of Region R in Problem 8

***Solution**:* Compute

$$
\begin{aligned}
\iint_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y &= \int_0^2 \int_0^{2x} e^{-x^2}\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^2 2xe^{-x^2}\ \mathrm{d}x \\
&= \left[-e^{-x^2}\right]_0^2 \\
&= 1 - e^{-4}.
\end{aligned}
$$

□