Review Problems for Final Exam

1. In this problem, u and v denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality

   $$|\ \|v\| - \|v\|\ | \leqslant \|v - u\| \quad \text{for all} \ \ u, v \in \mathbb{R}^n.$$

   (b) Use the inequality derived in the previous part to show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(v) = \|v\|$, for all $v \in \mathbb{R}^n$, is continuous in $\mathbb{R}^n$.

   (c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(v) = \sin(\|v\|)$, for all$v \in \mathbb{R}^n$, is continuous.

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|^2$ for all $v \in \mathbb{R}^n$.

   (a) Show that f is differentiable in $\mathbb{R}^n$ and compute the linear map

   $$Df(u)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \ \ u \in \mathbb{R}^n.$$

   What is the gradient of f at u for all $u \in \mathbb{R}^n$?

   (b) Let $\widehat{v}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector u in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|u - t\widehat{v}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

   (c) Let $\widehat{v}$ be as in the previous part. For any $u \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{v}$ which is the closest to u. Justify your answer.

3. Let I denote an open interval which contains the real number a. Assume that $\sigma\colon I \to \mathbb{R}^n$ is a C^1 parametrization of a curve C in $\mathbb{R}^n$. Define $s\colon I \to \mathbb{R}$ as follows:

   $$s(t) = \text{arlength along the curve } C \text{ from } \sigma(a) \text{ to } \sigma(t), \tag{1}$$

   for all $t \in I$.

   (a) Give a formula, in terms of an integral, for computing $s(t)$ for all $t \in I$.

   (b) Prove that s is differentiable on I and compute $s'(t)$ for all $t \in I$. Deduce that s is strictly increasing with increasing t.

   (c) Let $\ell =$ arclength of C, and suppose that $\gamma\colon [0, \ell] \to \mathbb{R}^n$ is a a parametrization of C with the arclength parameter s defined in (1); so that,

   $$C = \{\gamma(s) \mid 0 \leqslant s \leqslant \ell\}.$$

   Use the fact that $\sigma(t) = \gamma(s(t))$, for all $t \in [a, b]$, to show $\gamma'(s)$ is a unit vector that is tangent to the curve C at the point $\gamma(s)$.

4. Let I denote an open interval of real numbers and $f\colon I \to \mathbb{R}$ be a differentiable function. Let $a, b \in I$ be such that $a < b$, and define C to the section of the graph of $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$; that is,

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = f(x) \text{ and } a \leqslant x \leqslant b\}$$

   (a) By providing an appropriate parametrization of C, compute the arclenth of C, $\ell(C)$.

   (b) Let $f(x) = 5 - 2x^{3/2}$, for $x \geqslant 0$. Compute the exact arcength of $y = f(x)$ over the interval $[0, 11]$.

5. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

   (a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u, v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

   (b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

   (c) Evaluate the integral $\displaystyle\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

   (d) Evaluate the integral $\displaystyle\iint_T |\det[D\Phi(u, v)]|\, dudv$, where $\det[D\Phi(u, v)]$ denotes the determinant of the Jcobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).

6. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1 - y^2}\, dydx$.

   (a) Identify the region of integration, R, for this integral and sketch it.

   (b) Change the order of integration in the iterated integral and evaluate the double integral $\displaystyle\int_R x\sqrt{1 - y^2}\, dxdy$.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x, t) = f(x - ct) \quad \text{for all} \quad (x, t) \in \mathbb{R}^2,$$

where c is a real constant. Verify that $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

8. What is the region R over which you integrate when evaluating the iterated integral
$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}}\ \mathrm{d}y\ \mathrm{d}x?$$
Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define
$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$
   (a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.

   (b) Recall that the divergence of a vector field $F = P\,\widehat{\imath} + Q\,\widehat{\jmath}$ is the scalar field given by $\text{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Express the divergence of the gradient of u, in terms of f', f'' and r.
   The expression $\text{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.

10. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   (c) Find the largest and the smallest value of f on C.

11. Let ω be the differential 1–form in $\mathbb{R}^3$ given by $\omega = x\ dx + y\ dy + z\ dz$.

   (a) Compute the differential, $d\omega$, of ω.

   (b) If possible, find a differential 0–form, f, such that $\omega = df$.

   (c) Let C be parametrized by a C^1 connecting $P_o(1,-1,-2)$ to $P_1(-1,1,2)$. Compute the line integral $\displaystyle\int_C \omega$.

   (d) Let C denote any simple closed curve in $\mathbb{R}^3$. Evaluate the line integral $\displaystyle\int_C \omega$.

12. Let f denote a differential 0–form in $\mathbb{R}^3$ and ω a a differential 1–form in $\mathbb{R}^3$.

    (a) Verify that $d(df) = 0$.

    (b) Verify that $d(d\omega) = 0$.

13. Let f and g denote differential 0–forms in $\mathbb{R}^3$, and ω and η a differential 1–forms in $\mathbb{R}^3$. Derive the following identities

    (a) $d(fg) = g\ df + f\ dg$.

    (b) $d(f\omega) = df \wedge \omega + f\ d\omega$.

    (c) $d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta$.

14. Let R denote the square, $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, and ∂R denote the boundary of R oriented in the counterclockwise sense. Evaluate the line integral

$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy.$$