Solutions to Review Problems for Final Exam

1. In this problem, u and v denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality

   $$| \, \|v\| - \|u\| \, | \leqslant \|v - u\| \quad \text{for all} \quad u, v \in \mathbb{R}^n. \tag{1}$$

   ***Solution**:* Write

   $$\|u\| = \|(u - v) + v\|$$

   and applying the triangle inequality to obtain

   $$\|u\| \leqslant \|u - v\| + \|v\|,$$

   from which we get that

   $$\|u\| - \|v\| \leqslant \|v - u\|. \tag{2}$$

   Interchanging the roles for u and v in (2) we obtain

   $$\|v\| - \|u\| \leqslant \|u - v\|.$$

   from which we get

   $$\|v\| - \|u\| \leqslant \|v - u\|. \tag{3}$$

   Combining (2) and (3) yields

   $$-\|v - u\| \leqslant \|v\| - \|u\| \leqslant \|v - u\|,$$

   which is (1). □

   (b) Use the inequality derived in the previous part to show that the function $f \colon \mathbb{R}^n \to \mathbb{R}$ given by $f(v) = \|v\|$, for all $v \in \mathbb{R}^n$, is continuous in $\mathbb{R}^n$.

   ***Solution**:* Fix $u \in \mathbb{R}^n$ and apply the inequality in (1) to any $v \in \mathbb{R}^n$ to obtain that

   $$|\|v\| - \|u\|| \leqslant \|v - u\|,$$

   or

   $$|f(v) - f(u)| \leqslant \|v - u\|. \tag{4}$$

   Next, apply the Squeeze Lemma to obtain from (4) that

   $$\lim_{\|v-u\| \to 0} |f(v) - f(u)| = 0,$$

   which shows that f is continuous at u for any $u \in \mathbb{R}^n$. □

(c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(v) = \sin(\|v\|)$, for all$v \in \mathbb{R}^n$, is continuous.

Solution: Observe that $g = \sin \circ f$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is as defined in part (b). Thus, g is the composition of two continuous functions, and is, therefore, continuous. □

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|^2$ for all $v \in \mathbb{R}^n$.

(a) Show that f is differentiable in $\mathbb{R}^n$ and compute the linear map

$$Df(u)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \quad u \in \mathbb{R}^n.$$

What is the gradient of f at u for all $u \in \mathbb{R}^n$?

Solution: Let $u \in \mathbb{R}^n$ and compute

$$\begin{aligned} f(u+w) &= \|u+w\|^2 \\ &= (u+w)\cdot(u+w) \\ &= u\cdot u + u\cdot w + w\cdot u + w\cdot w \\ &= \|u\|^2 + 2u\cdot w + \|w\|^2, \end{aligned}$$

for $w \in \mathbb{R}^n$, where we have used the symmetry of the dot product and the fact that $\|v\|^2 = v\cdot v$ for all $v \in \mathbb{R}^n$. We therefore have that

$$f(u+w) = f(u) + 2u\cdot w + \|w\|^2, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n. \tag{5}$$

Writing

$$Df(u)w = 2u\cdot w, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{6}$$

and

$$E_u(w) = \|w\|^2, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{7}$$

we see that (5) can be rewritten as

$$f(u+w) = f(u) + Df(u)w + E_u(w), \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{8}$$

where, according to (6), $Df(u)\colon \mathbb{R}^n \to \mathbb{R}^n$ defines a linear transformation, and, by virtue of (7),

$$\frac{|E_u(w)|}{\|w\|} = \|w\|, \qquad \text{for } w \neq 0,$$

from which we get that

$$\lim_{\|w\|\to 0} \frac{|E_u(w)|}{\|w\|} = 0.$$

Consequently, in view of (8), we conclude that f is differentiable at every $u \in \mathbb{R}^n$, derivative at u given by (6).

Since $Df(u)w = \nabla f(u) \cdot w$, for all u and w in $\mathbb{R}^n$, by comparing with (6), we see that

$$\nabla f(u) = 2u, \qquad \text{for all } u \in \mathbb{R}^n.$$

□

Alternate Solution: Alternatively, for $u = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have that

$$f(u) = x_1^2 + x_2^2 + \cdots + x_n^2;$$

so that

$$\frac{\partial f}{\partial x_j}(u) = 2x_j, \qquad \text{for } j = 1, 2, \ldots, n.$$

Thus, all the partial derivatives of f are continuous on $\mathbb{R}^n$; that is, f is a C^1 function. Consequently, f is differentiable on $\mathbb{R}^n$. Furthermore,

$$Df(u)w = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \end{pmatrix} w, \qquad \text{for all } w \in \mathbb{R}^n,$$

which can be written as

$$Df(u)w = 2u \cdot w, \qquad \text{for all } w \in \mathbb{R}^n. \tag{9}$$

It then follows that $\nabla f(u) = 2u$ for all $u \in \mathbb{R}^n$. □

(b) Let $\widehat{v}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector u in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|u - t\widehat{v}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

Solution: Observe that $g = f \circ \sigma$, where $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ is given by

$$\sigma(t) = u - t\widehat{v}, \qquad \text{for all } t \in \mathbb{R}. \tag{10}$$

Thus, σ is a differentiable path with

$$\sigma'(t) = -\widehat{v}, \qquad \text{for all } t \in \mathbb{R}. \tag{11}$$

Thus, by the result from part (a), g is the composition of two differentiable functions. Consequently, by the Chain Rule, g is differentiable with

$$g'(t) = Df(\sigma(t))\sigma'(t), \qquad \text{for all } t \in \mathbb{R}. \tag{12}$$

Thus, using (9) and (11), we obtain from (12) that

$$g'(t) = 2\sigma(t) \cdot (-\widehat{v}), \quad \text{for all } t \in \mathbb{R},$$

or

$$g'(t) = -2\sigma(t) \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R}. \tag{13}$$

Thus, using (10), we obtain from (13) that

$$g'(t) = -2(u - t\widehat{v}) \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R},$$

which leads to

$$g'(t) = 2t - 2u \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R}, \tag{14}$$

since $\widehat{v}$ is a unit vector in $\mathbb{R}^n$. □

(c) Let $\widehat{v}$ be as in the previous part. For any $u \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{v}$ which is the closest to u. Justify your answer.

***Solution**:* It follows from (14) that $g''(t) = 2 > 0$ for all $t \in \mathbb{R}$; so that g has a global minimum when $g'(t) = 0$. We then obtain from (14) that $g(t)$ is the smallest possible when

$$t = u \cdot \widehat{v}.$$

Consequently, the point on the line spanned by $\widehat{v}$ which is the closest to u is $(u \cdot \widehat{v})\widehat{v}$, or the orthogonal projection of u onto the direction of $\widehat{v}$. □

3. Let I denote an open interval which contains the real number a. Assume that $\sigma \colon I \to \mathbb{R}^n$ is a C^1 parametrization of a curve C in $\mathbb{R}^n$. Define $s \colon I \to \mathbb{R}$ as follows:

$$s(t) = \text{arlength along the curve } C \text{ from } \sigma(a) \text{ to } \sigma(t), \tag{15}$$

for all $t \in I$.

(a) Give a formula, in terms of an integral, for computing $s(t)$ for all $t \in I$.

***Answer**:*

$$s(t) = \int_a^t \|\sigma'(\tau)\| \, d\tau, \quad \text{for all } t \in I. \tag{16}$$

□

(b) Prove that s is differentiable on I and compute $s'(t)$ for all $t \in I$. Deduce that s is strictly increasing with increasing t.

***Solution**:* It follows from the assumption that σ is C^1, the Fundamental Theorem of Calculus, and (17), that s is differentiable and

$$s'(t) = \|\sigma'(t)\|, \quad \text{for all } t \in I. \tag{17}$$

Since we are also assuming that σ is a parametrization of a C^1 curve, C, it follows that $\sigma'(t) \neq \mathbf{0}$ for all $t \in I$. Consequently, we obtain from (17) that

$$s'(t) > 0, \quad \text{for all } t \in I,$$

which shows that $s(t)$ is strictly increasing with increasing t. □

(c) Let $\ell =$ arclength of C, and suppose that $\gamma\colon [0, \ell] \to \mathbb{R}^n$ is a a parametrization of C with the arclength parameter s defined in (15); so that,

$$C = \{\gamma(s) \mid 0 \leqslant s \leqslant \ell\}.$$

Use the fact that $\sigma(t) = \gamma(s(t))$, for all $t \in [a, b]$, to show $\gamma'(s)$ is a unit vector that is tangent to the curve C at the point $\gamma(s)$.

***Solution**:* Note that $\sigma = \gamma \circ s$ is a composition of two differentiable functions, by the result of part (b). Consequently, by the Chain Rule,

$$\sigma'(t) = \frac{ds}{dt}\gamma'(s), \quad \text{for } t \in (a, b).$$

Thus, using (17),

$$\sigma'(t) = \|\sigma'(t)\|\gamma'(s), \quad \text{for } t \in (a, b).$$

So, using the fact that $\|\sigma'(t)\| > 0$ for all $t \in (a, b)$,

$$\gamma'(s) = \frac{1}{\|\sigma'(t)\|}\, \sigma'(t), \quad \text{for } t \in (a, b),$$

which shows that $\gamma'(s)$ is a unit vector that is tangent to the curve C at the point $\gamma(s)$. □

4. Let I denote an open interval of real numbers and $f\colon I \to \mathbb{R}$ be a differentiable function. Let $a, b \in I$ be such that $a < b$, and define C to the section of the graph of $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$; that is,

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = f(x) \text{ and } a \leqslant x \leqslant b\}$$

   (a) By providing an appropriate parametrization of C, compute the arclenth of C, $\ell(C)$.

   ***Solution**:* Parametrize C by $\sigma\colon [a, b] \to \mathbb{R}^2$ given by

$$\sigma(t) = (t, f(t)), \qquad \text{for } a \leqslant t \leqslant b.$$

   Then,

$$\sigma'(t) = (1, f'(t)), \qquad \text{for } a \leqslant t \leqslant b;$$

   so that

$$\|\sigma'(t)\| = \sqrt{1 + [f'(t)]^2}, \qquad \text{for } a \leqslant t \leqslant b.$$

   Therefore,

$$\ell(C) = \int_a^b \sqrt{1 + [f'(t)]^2}\; dt. \tag{18}$$

□

   (b) Let $f(x) = 5 - 2x^{3/2}$, for $x \geqslant 0$. Compute the exact arcength of $y = f(x)$ over the interval $[0, 11]$.

   ***Solution**:* We use the formula in (18) with

$$f'(t) = -3t^{1/2}, \qquad \text{for } t > 0.$$

   Thus,

$$\begin{aligned} \ell(C) &= \int_0^{11} \sqrt{1 + 9t}\; dt \\ &= \left[\frac{2}{27}(1 + 9t)^{3/2}\right]_0^{11} \\ &= \frac{2}{27}(1000 - 1) \\ &= 74. \end{aligned}$$

□

5. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

(a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u,v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

***Solution**:* Write

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

where $f(u,v) = 2u$ and $g(u,v) = v^2$ for all $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$. Observe that the partial derivatives of f and g exist and are given by

$$\frac{\partial f}{\partial u}(u,v) = 2, \qquad \frac{\partial f}{\partial v}(u,v) = 0$$

$$\frac{\partial g}{\partial u}(u,v) = 0, \qquad \frac{\partial g}{\partial v}(u,v) = 2v.$$

Note that the partial derivatives of f and g are continuous. Therefore, Φ is a C^1 map. Hence, Φ is differentiable on $\mathbb{R}^2$ and its derivative map at (u,v), for any $(u,v) \in \mathbb{R}^2$, is given by multiplication by the Jacobian matrix

$$D\Phi(u,v) = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix};$$

that is,

$$D\Phi(u,v)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2h \\ 2vk \end{pmatrix}$$

for all $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{R}^2$. □

(b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

***Solution**:* The image of T under Φ is the set

$$\begin{aligned} \Phi(T) &= \{(x,y) \in \mathbb{R}^2 \mid x = 2u, y = v^2, \text{ for some } (u,v) \in \mathbb{R}\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x^2/4\}. \end{aligned}$$

A sketch of $R = \Phi(T)$ is shown in Figure 1. □

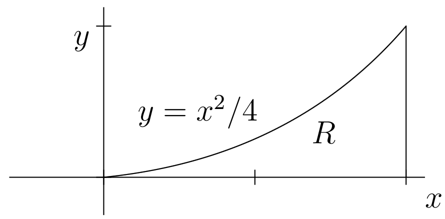

Figure 1: Sketch of Region $\Phi(T)$

(c) Evaluate the integral $\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

***Solution**:* Compute by means of iterated integrals

$$\begin{aligned}\iint_R dxdy &= \int_0^2 \int_0^{x^2/4} dy\, dx \\ &= \int_0^2 \frac{x^2}{4}\, dx \\ &= \left[\frac{x^3}{12}\right]_0^2 \\ &= \frac{2}{3}.\end{aligned}$$

□

(d) Evaluate the integral $\iint_T |\det[D\Phi(u,v)]|\, dudv$, where $\det[D\Phi(u,v)]$ denotes the determinant of the Jacobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).

***Solution**:* Compute $\det[D\Phi(u,v)]$ to get

$$\det[D\Phi(u,v)] = 4v.$$

so that

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4|v|\, dudv,$$

where the region T, in the uv–plane is sketched in Figure 2. Observe that,

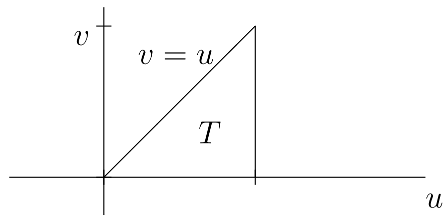

Figure 2: Sketch of Region T

in that region, $v \geqslant 0$, so that

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4v \ dudv,$$

Compute by means of iterated integrals

$$\begin{aligned}\iint_T |\det[D\Phi(u,v)]| dudv &= \int_0^1 \int_0^u 4v \ dvdu \\ &= \int_0^1 2u^2 \ du \\ &= \frac{2}{3},\end{aligned}$$

which is the same result as that obtained in part (c). □

6. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2} \ dydx$.

   (a) Identify the region of integration, R, for this integral and sketch it.
   ***Solution**:* The region $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant 1, \ 0 \leqslant x \leqslant 1\}$ is sketched in Figure 3. □

   (b) Change the order of integration in the iterated integral and evaluate the double integral $\displaystyle\int_R x\sqrt{1-y^2} \ dxdy$.

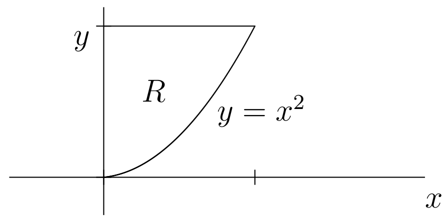

Figure 3: Sketch of Region R

Solution: Compute

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\ dxdy &= \int_0^1 \int_0^{\sqrt{y}} x\sqrt{1-y^2}\ dxdy \\ &= \int_0^1 \left[\frac{x^2}{2}\sqrt{1-y^2}\right]_0^{\sqrt{y}}\ dy \\ &= \int_0^1 \frac{y}{2}\sqrt{1-y^2}\ dy.\end{aligned}$$

Next, make the change of variables $u = 1 - y^2$ to obtain that

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\ dxdy &= -\frac{1}{4}\int_1^0 \sqrt{u}\ du \\ &= \frac{1}{4}\int_0^1 \sqrt{u}\ du \\ &= \frac{1}{6}.\end{aligned}$$

□

7. What is the region R over which you integrate when evaluating the iterated integral

$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}}\ \mathrm{d}y\ \mathrm{d}x?$$

Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

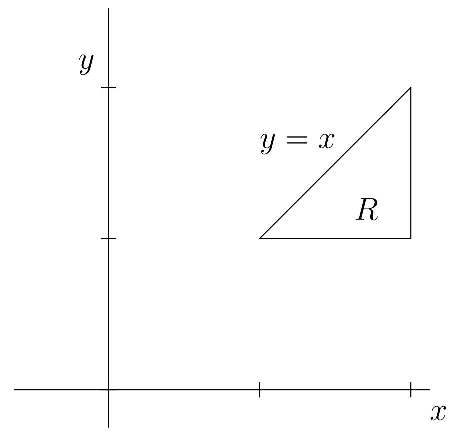

Figure 4: Sketch of Region R

Solution: The region $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant y \leqslant x,\ 1 \leqslant x \leqslant 2\}$ is sketched in Figure 4. Interchanging the order of integration, we obtain that

$$\iint_R \frac{x}{\sqrt{x^2 + y^2}}\, dxdy = \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2 + y^2}}\, dxdy. \tag{19}$$

The iterated integral in (19) is easier to evaluate; in fact,

$$\begin{aligned} \iint_R \frac{x}{\sqrt{x^2 + y^2}}\, dxdy &= \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2 + y^2}}\, dxdy \\ &= \int_1^2 \left[\sqrt{x^2 + y^2}\right]_y^2 dy \\ &= \int_1^2 \left[\sqrt{4 + y^2} - \sqrt{2}\, y\right] dy. \end{aligned}$$

We therefore get that

$$\iint_R \frac{x}{\sqrt{x^2 + y^2}}\, dxdy = \int_1^2 \sqrt{4 + y^2}\, dy - \sqrt{2} \int_1^2 y\, dy. \tag{20}$$

Evaluating the second integral on the right–hand side of (20) yields

$$\int_1^2 y\, dy = \frac{3}{2}. \tag{21}$$

The first integral on the right–hand side of (20) leads to

$$\int_1^2 \sqrt{4+y^2}\; dy \;=\; \left[\frac{y}{2}\sqrt{4+y^2} + \frac{4}{2}\ln\left|y+\sqrt{4+y^2}\right|\right]_1^2,$$

which evaluates to

$$\int_1^2 \sqrt{4+y^2}\; dy \;=\; 2\sqrt{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right). \qquad (22)$$

Substituting (21) and (22) into (20) we obtain

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\; dxdy \;=\; \frac{\sqrt{2}}{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right).$$

□

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x-ct) \quad \text{for all} \;\; (x,t) \in \mathbb{R}^2,$$

where c is a real constant.

Verify that $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$.

***Solution**:* Apply the Chain Rule to obtain

$$\frac{\partial u}{\partial x} = f'(x-ct)\cdot\frac{\partial}{\partial x}(x-ct) = f'(x-ct).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct), \qquad (23)$$

$$\frac{\partial u}{\partial t} = f'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = -cf'(x-ct),$$

and

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct). \qquad (24)$$

Combining (23) and (24) we see that

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct) = c^2\frac{\partial^2 u}{\partial x^2},$$

which was to be verified. □

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

(a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.

***Solution**:* Compute

$$F(x,y) = \nabla u(x,y) = \frac{\partial u}{\partial x}\,\widehat{i} + \frac{\partial u}{\partial y}\,\widehat{j}, \tag{25}$$

where, by the Chain Rule,

$$\frac{\partial u}{\partial x} = f'(r)\,\frac{\partial r}{\partial x} \tag{26}$$

and

$$\frac{\partial u}{\partial y} = f'(r)\,\frac{\partial r}{\partial y}. \tag{27}$$

In order to compute $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial r}{\partial x}$, write

$$r^2 = x^2 + y^2, \tag{28}$$

and differentiate with respect to x on both sides of (28) to obtain

$$2r\frac{\partial r}{\partial x} = 2x,$$

from which we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \text{for } (x,y) \neq (0,0). \tag{29}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \text{for } (x,y) \neq (0,0). \tag{30}$$

Substituting (29) into (26) yields

$$\frac{\partial u}{\partial x} = \frac{f'(r)}{r}\,x. \tag{31}$$

Similarly, substituting (30) into (27) yields

$$\frac{\partial u}{\partial y} = \frac{f'(r)}{r}\,y. \tag{32}$$

Next, substitute (31) and (32) into (25) to obtain

$$F(x,y) = \frac{f'(r)}{r}(x\,\widehat{i} + y\,\widehat{j}), \tag{33}$$

□

(b) Recall that the divergence of a vector field $F = P\,\widehat{i} + Q\,\widehat{j}$ is the scalar field given by $\text{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Express the divergence of the gradient of u, in terms of f', f'' and r.

The expression $\text{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.

***Solution**:* From (33) we obtain that

$$P(x,y) = \frac{f'(r)}{r}\,x \quad \text{and} \quad Q(x,y) = \frac{f'(r)}{r}\,y,$$

so that, applying the Product Rule, Chain Rule and Quotient Rule,

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{f'(r)}{r} + x\frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial x} \\ &= \frac{f'(r)}{r} + x\,\frac{rf''(r) - f'(r)}{r^2}\,\frac{x}{r}, \end{aligned} \tag{34}$$

where we have also used (29). Simplifying the expression in (34) yields

$$\frac{\partial P}{\partial x} = \frac{f'(r)}{r} + x^2\,\frac{f''(r)}{r^2} - x^2\,\frac{f'(r)}{r^3}. \tag{35}$$

Similar calculations lead to

$$\frac{\partial Q}{\partial y} = \frac{f'(r)}{r} + y^2\,\frac{f''(r)}{r^2} - y^2\,\frac{f'(r)}{r^3}. \tag{36}$$

Adding the results in (35) and (36), we then obtain that

$$\begin{aligned} \text{div}F &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= 2\,\frac{f'(r)}{r} + r^2\,\frac{f''(r)}{r^2} - r^2\,\frac{f'(r)}{r^3}, \end{aligned} \tag{37}$$

where we have used (28). Simplifying the expression in (37), we get that

$$\text{div}F = f''(r) + \frac{f'(r)}{r}.$$

□

10. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

    (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.
    ***Solution**:* The curve C is the graph of the equation

$$\frac{x^2}{1/2} + y^2 = 1,$$

which is sketched in Figure 5. □

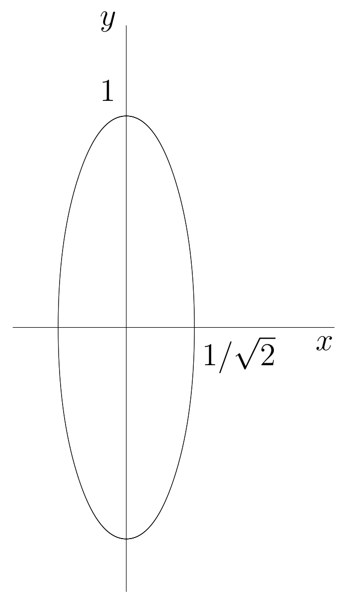

Figure 5: Sketch of ellipse

    (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.
    ***Solution**:* Let $\sigma\colon [0, 2\pi] \to \mathbb{R}^2$ denote the C^1 parametrization of C given by

$$\sigma(t) = \left(\frac{\sqrt{2}}{2}\cos t,\ \sin t\right), \qquad \text{for all } t \in [0, 2\pi].$$

We then have that

$$g(\sigma(t)) = 1, \qquad \text{for all } t. \tag{38}$$

Differentiating on both sides of (38) yields that

$$\nabla g(\sigma(t)) \cdot \sigma'(t) = 0, \qquad \text{for all } t,$$

where we have applied the Chain Rule, which shows that $\nabla g(x, y)$ is perpendicular to the tangent vector to C at (x, y).

Next, suppose that $f(\sigma(t))$ has a critical point at t_o. Then, the derivative of $f(\sigma(t))$ at t_o is 0; that is,

$$\nabla f(\sigma(t_o)) \cdot \sigma'(t_0) = 0,$$

where we have applied the Chain Rule. It then follows that $\nabla f(x_o, y_o)$ is perpendicular to the tangent vector to C at a critical point (x_o, y_o). Hence, $\nabla f(x_o, y_o)$ must be parallel to $\nabla g(x_o, y_o)$. □

(c) Find the largest and the smallest value of f on C.

Solution*:* By the result of part (b), at a critical point, (x, y), of f on C, it must be the case that

$$\nabla g(x, y) = \lambda \nabla f(x, y), \tag{39}$$

for some non–zero real number λ, where

$$\nabla f(x, y) = 4\,\widehat{\imath} - 7\,\widehat{\jmath}, \tag{40}$$

and

$$\nabla g(x, y) = 4x\,\widehat{\imath} + 2y\,\widehat{\jmath}. \tag{41}$$

Substituting (40) and (41) into (39) yields the pair of equations

$$x = \lambda \tag{42}$$

and

$$2y = -7\lambda. \tag{43}$$

Substituting the expressions for x and y in (42) and (43), respectively, into the equation of the ellipse

$$2x^2 + y^2 = 1,$$

yields that

$$\frac{57}{4}\,\lambda^2 = 1,$$

from which we get that

$$\lambda = \pm\frac{2\sqrt{57}}{57}. \tag{44}$$

The values for λ in (44), together with (42) and (43), yield the critical points

$$\left(\frac{2\sqrt{57}}{57}, -\frac{7\sqrt{57}}{57}\right) \quad \text{and} \quad \left(-\frac{2\sqrt{57}}{57}, \frac{7\sqrt{57}}{57}\right). \tag{45}$$

Evaluating the function f at each of the critical points in (45) we obtain that

$$f\left(\frac{2\sqrt{57}}{57}, -\frac{7\sqrt{57}}{57}\right) = \sqrt{57} \quad \text{and} \quad f\left(-\frac{2\sqrt{57}}{57}, \frac{7\sqrt{57}}{57}\right) = -\sqrt{57}.$$

Consequently, the largest value of f on C is $\sqrt{57}$ and the smallest value is $-\sqrt{57}$. □

11. Let ω be the differential 1–form in $\mathbb{R}^3$ given by $\omega = x\ dx + y\ dy + z\ dz$.

(a) Compute the differential, $d\omega$, of ω.
***Solution**:* Compute $d\omega = dx \wedge dx + dy \wedge dy + dz \wedge dz = 0$. □

(b) If possible, find a differential 0–form, f, such that $\omega = df$.
***Solution**:* Let $f(x, y, z) = \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2}$. Then,

$$df = x\ dx + y\ dy + z\ dz = \omega.$$

□

(c) Let C be parametrized by a C^1 connecting $P_o(1, -1, -2)$ to $P_1(-1, 1, 2)$. Compute the line integral $\displaystyle\int_C \omega$.
***Solution**:* Apply the Fundamental Theorem of Calculus,

$$\int_C \omega = \int_C df = f(P_1) - f(P_o),$$

where f is as given in part (b). Consequently,

$$\int_C \omega = f(-1, 1, 2) - f(1, -1, -2) = 3 - 3 = 0.$$

□

(d) Let C denote any simple closed curve in $\mathbb{R}^3$. Evaluate the line integral $\int_C \omega$.

Solution: $\int_C \omega = 0$, since C is closed and ω is exact. □

12. Let f denote a differential 0–form in $\mathbb{R}^3$ and ω a a differential 1–form in $\mathbb{R}^3$.

(a) Verify that $d(df) = 0$.

Solution: Compute

$$
\begin{aligned}
d(df) &= d\left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz\right) \\
&= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz \\
&= \frac{\partial^2 f}{\partial x^2}\, dx \wedge dx + \frac{\partial^2 f}{\partial y \partial x}\, dy \wedge dx + \frac{\partial^2 f}{\partial z \partial x}\, dz \wedge dx \\
&\quad + \frac{\partial^2 f}{\partial x \partial y}\, dx \wedge dy + \frac{\partial^2 f}{\partial y^2}\, dy \wedge dy + \frac{\partial^2 f}{\partial z \partial y}\, dz \wedge dy \\
&\quad + \frac{\partial^2 f}{\partial x \partial z}\, dx \wedge dz + \frac{\partial^2 f}{\partial y \partial z}\, dy \wedge dz + \frac{\partial^2 f}{\partial z^2}\, dz \wedge dz,
\end{aligned}
$$

so that

$$
\begin{aligned}
d(df) &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) dy \wedge dz \\
&\quad + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) dz \wedge dx \\
&\quad + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) dx \wedge dy.
\end{aligned} \tag{46}
$$

It follows from (46) and the fact that the mixed second partial derivatives of a C^∞ function are equal that $d(df) = 0$. □

(b) Verify that $d(d\omega) = 0$.

Solution: Since ω is a differential 1–form, we may write

$$\omega = f_1\, dx + f_2\, dy + f_3\, dz,$$

where f_1, f_2 and f_3 are differential 0–forms. Thus, since the operator d is linear

$$d(d\omega) = d(df_1) \wedge dx + d(df_2) \wedge dy + d(df_3) \wedge dz,$$

which is 0 since $d(df_j) = 0$, for $j = 1, 2, 3$, by the result from part (a). Hence, $d(d\omega) = 0$, which was to be shown. □

13. Let f and g denote differential 0–forms in $\mathbb{R}^3$, and ω and η a differential 1–forms in $\mathbb{R}^3$. Derive the following identities

(a) $d(fg) = g\ df + f\ dg$.

***Solution**:* Compute

$$\begin{aligned} d(fg) &= \frac{\partial(fg)}{\partial x}\ dx + \frac{\partial(fg)}{\partial y}\ dy + \frac{\partial(fg)}{\partial z}\ dz \\ &= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) dx \\ &\quad + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) dy \\ &\quad + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right) dz, \end{aligned} \tag{47}$$

where we have used the Product Rule. Rearranging terms in (47) we obtain that

$$\begin{aligned} d(fg) &= g\left(\frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz\right) \\ &\quad f\left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y}\ dy + \frac{\partial g}{\partial z}\ dz\right) \\ &= g\ df + f\ dg. \end{aligned}$$

□

(b) $d(f\omega) = df \wedge \omega + f\ d\omega$.

***Solution**:* Write $\omega = g_1\ dx + g_2\ dy + g_3\ dz$, where g_1, g_2 and g_3 are differential 0–forms. We then have that

$$f\omega = fg_1\ dx + fg_2\ dy + fg_3\ dz,$$

so that

$$d(f\omega) = d(fg_1) \wedge dx + d(fg_2) \wedge dy + d(fg_3) \wedge dz. \tag{48}$$

Applying the result from part (a), we obtain from (48) that

$$\begin{aligned} d(f\omega) &= (f dg_1 + g_1 df)\wedge dx + (f dg_2 + g_2 df)\wedge dy \\ &\quad +(f dg_3 + g_3 df)\wedge dz \\ &= f\ dg_1\wedge dx + g_1\ df\wedge dx \\ &\quad +f\ dg_2\wedge dy + g_2\ df\wedge dy \\ &\quad +f\ dg_3\wedge dz + g_3\ df\wedge dz \\ &= f(dg_1\wedge dx + dg_2\wedge dy + dg_3\wedge dz) \\ &\quad +df\wedge g_1\ dx + df\wedge g_2\ dy + df\wedge g_3\ dz, \end{aligned} \tag{49}$$

where we have used the bi–linearity of the wedge product. Using bi–linearity again, we obtain from (49) that

$$\begin{aligned} d(f\omega) &= f\ d\omega + df\wedge(g_1\ dx + g_2\ dy + g_3\ dz) \\ &= f\ d\omega + df\wedge\omega, \end{aligned}$$

which was to be shown. □

(c) $d(\omega\wedge\eta) = d\omega\wedge\eta - \omega\wedge d\eta$.

***Solution**:* Write $\omega = f_1\ dx + f_2\ dy + f_3\ dz$, where f_1, f_2 and f_3 are differential 0–forms, and compute

$$\begin{aligned} \omega\wedge\eta &= (f_1\ dx + f_2\ dy + f_3\ dz)\wedge\eta \\ &= f_1\ dx\wedge\eta + f_2\ dy\wedge\eta + f_3\ dz\wedge\eta, \end{aligned}$$

so that

$$d(\omega\wedge\eta) = d(f_1\ dx\wedge\eta) + d(f_2\ dy\wedge\eta) + d(f_3\ dz\wedge\eta). \tag{50}$$

Using the result from part (b), we obtain from (50) that

$$\begin{aligned} d(\omega\wedge\eta) &= f_1\ d(dx\wedge\eta) + df_1\wedge dx\wedge\eta \\ &\quad +f_2\ d(dy\wedge\eta) + df_2\wedge dy\wedge\eta \\ &\quad +f_3\ d(dz\wedge\eta) + df_3\wedge dz\wedge\eta, \end{aligned} \tag{51}$$

where we have used the associativity of the wedge product. Next, write $\eta = g_1\ dx + g_2\ dy + g_3\ dz$, where g_1, g_2 and g_3 are differential 0–forms. Then,

$$dx \wedge \eta = g_2\ dx \wedge dy + g_3\ dx \wedge dz,$$

so that

$$\begin{aligned} d(dx \wedge \eta) &= dg_2 \wedge dx \wedge dy + dg_3 \wedge dx \wedge dz \\ &= \frac{\partial g_2}{\partial z}\ dz \wedge dx \wedge dy + \frac{\partial g_3}{\partial y}\ dy \wedge dx \wedge dz \\ &= \left(\frac{\partial g_2}{\partial z} - \frac{\partial g_3}{\partial y}\right) dx \wedge dy \wedge dz, \end{aligned} \tag{52}$$

where we have used the anti–commutativity of the wedge product. On the other hand, note that

$$\begin{aligned} d\eta &= dg_1 \wedge dx + dg_2 \wedge dy + dg_3 \wedge dz \\ &= \left(\frac{\partial g_1}{\partial x}\ dx + \frac{\partial g_1}{\partial y}\ dy + \frac{\partial g_1}{\partial z}\ dz\right) \wedge dx \\ &\quad + \left(\frac{\partial g_2}{\partial x}\ dx + \frac{\partial g_2}{\partial y}\ dy + \frac{\partial g_2}{\partial z}\ dz\right) \wedge dy \\ &\quad + \left(\frac{\partial g_3}{\partial x}\ dx + \frac{\partial g_3}{\partial y}\ dy + \frac{\partial g_3}{\partial z}\ dz\right) \wedge dz, \end{aligned}$$

so that, using the anti–commutativity of the wedge product,

$$\begin{aligned} d\eta &= \frac{\partial g_1}{\partial y}\, dy\wedge dx + \frac{\partial g_1}{\partial z}\, dz\wedge dx \\ &\quad + \frac{\partial g_2}{\partial x}\, dx\wedge dy + \frac{\partial g_2}{\partial z}\, dz\wedge dy \\ &\quad + \frac{\partial g_3}{\partial x}\, dx\wedge dz + \frac{\partial g_3}{\partial y}\, dy\wedge dz \\ &= \frac{\partial g_3}{\partial y}\, dy\wedge dz + \frac{\partial g_2}{\partial z}\, dz\wedge dy \\ &\quad + \frac{\partial g_3}{\partial x}\, dx\wedge dz + \frac{\partial g_1}{\partial z}\, dz\wedge dx \\ &\quad + \frac{\partial g_2}{\partial x}\, dx\wedge dy + \frac{\partial g_1}{\partial y}\, dy\wedge dx \\ &= \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) dy\wedge dz \\ &\quad + \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) dz\wedge dx \\ &\quad + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) dx\wedge dy, \end{aligned}$$

so that

$$\begin{aligned} d\eta &= \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) dy\wedge dz \\ &\quad + \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) dz\wedge dx \qquad (53) \\ &\quad + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) dx\wedge dy, \end{aligned}$$

Taking the wedge product with dx on the left of (53) yields

$$dx\wedge d\eta = \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) dx\wedge dy\wedge dz, \qquad (54)$$

Comparison of (52) and (54) yields the identity

$$d(dx \wedge \eta) \;=\; -dx \wedge d\eta \tag{55}$$

Similar calculations using (53) yield the additional identities

$$d(dy \wedge \eta) \;=\; -dy \wedge d\eta \tag{56}$$

and

$$d(dz \wedge \eta) \;=\; -dz \wedge d\eta. \tag{57}$$

Next, substitute the identities in (55), (56) and (57) into (51) to obtain

$$\begin{aligned} d(\omega \wedge \eta) \;=\; & -f_1\ (dx \wedge d\eta) + df_1 \wedge dx \wedge \eta \\ & -f_2\ (dy \wedge d\eta) + df_2 \wedge dy \wedge \eta \\ & -f_3\ (dz \wedge d\eta) + df_3 \wedge dz \wedge \eta, \end{aligned}$$

which leads to

$$\begin{aligned} d(\omega \wedge \eta) \;=\; & -(f_1\ dx + f_2\ dy + f_3\ dz) \wedge d\eta) \\ & +(df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz) \wedge \eta, \end{aligned} \tag{58}$$

by virtue of the bi–linearity of the wedge product. We therefore obtain from (58) that

$$d(\omega \wedge \eta) \;=\; -\omega \wedge d\eta + d\omega \wedge \eta,$$

which was to be shown. □

14. Let R denote the square, $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, and ∂R denote the boundary of R oriented in the counterclockwise sense. Evaluate the line integral

$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy.$$

***Solution**:* Apply the Fundamental Theorem of Calculus to get

$$\begin{aligned}
\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy &= \int_R d[(y^2 + x^3)\ dx + x^4\ dy] \\
&= \int_R 2y\ dy + 3x^2\ dx) \wedge dx + 4x^3\ dx \wedge dy \\
&= \int_R 2y\ dy \wedge dx + 4x^3\ dx \wedge dy \\
&= \int_R (4x^3 - 2y)\ dx \wedge dy,
\end{aligned}$$

so that

$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy = \iint_R (4x^3 - 2y)\ dxdy, \qquad (59)$$

since ∂R is oriented in the counterclockwise sense. Evaluating the double integral in (59) we obtain that

$$\begin{aligned}
\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy &= \int_0^1 \int_0^1 (4x^3 - 2y)\ dxdy \\
&= \int_0^1 \left[x^4 - 2xy\right]_0^1 dy \\
&= \int_0^1 (1 - 2y) dy \\
&= \left[y - y^2\right]_0^1 \\
&= 0.
\end{aligned}$$

□