Assignment #4

Due on Friday, October 4, 2019

Read Section 4.1, *Gâteaux Differentiability,* in the class lecture notes at
http://pages.pomona.edu/~ajr04747/

Background and Definitions

Gâteaux Differentiability. Let V denote a normed linear space, and V_o a non–trivial subspace of V. Let $J\colon V \to \mathbb{R}$ be a functional defined on V. We say that J is Gâteaux differentiable at $u \in V$ in the direction of $v \in V_o$ if

$$\frac{d}{dt} J(u + tv)\Big|_{t=0} = \lim_{t\to 0} \frac{J(u + tv) - J(u)}{t} \quad \text{exists.}$$

If the limit exists, we denote it by $dJ(u; v)$, and call it the Gâteaux derivative of J at u in the direction of v, or the first variation of J at u in the direction of v.

Do the following problems

1. Let $V = C^1([a, b], \mathbb{R})$ and $V_o = C^1_o([a, b], \mathbb{R})$. Define

$$J(y) = \frac{1}{2} \int_a^b (y'(x))^2 \, dx, \qquad \text{for all } y \in C^1([a, b], \mathbb{R}).$$

   Show that $J\colon V \to \mathbb{R}$ is Gâteaux differentiable at every $y \in V$ in the direction of $v \in V_o$, and compute $dJ(y; v)$ for all $y \in V$ and $v \in V_o$.

2. Let $G : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $J : C^1([a, b], \mathbb{R}) \to \mathbb{R}$ be the functional given by

$$J(y) = \int_a^b x^2 (y'(x))^3 \, \mathrm{d}x + G(y(b)) \quad \text{for all } y \in C^1([a, b], \mathbb{R}).$$

   Show that J is Gâteaux differentiable at every $y \in C^1([a, b], \mathbb{R})$ in the direction of $\eta \in C^1_o([a, b], \mathbb{R})$ and compute $dJ(y; \eta)$ for all $y \in C^1([a, b], \mathbb{R})$ and $\eta \in C^1_o([a, b], \mathbb{R})$.

3. Let V be a normed linear space and $L\colon V \to \mathbb{R}$ be a linear function. Prove that L is Gâteaux differentiable in V in the direction of any $v \in V$ and compute $dL(u, ; v)$ for any $u \in V$ and $v \in V$.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a C^1 real valued function of a single variable, and define $J\colon C^1([a,b],\mathbb{R}) \to \mathbb{R}$ by $J(y) = \displaystyle\int_a^b f(y'(x))\; \mathrm{d}x$ for all $y \in C^1([a,b],\mathbb{R})$. Show that J is Gâteaux differentiable at every $y \in C^1([a,b],\mathbb{R})$ in the direction of every $\eta \in C_o^1([a,b],\mathbb{R})$, and compute the Gâteaux derivative of J at every $y \in C^1([a,b],\mathbb{R})$ in the direction of every $\eta \in C_o^1([a,b],\mathbb{R})$.

5. Let V denote a normed linear space and V_o a non–trivial subspace of V. Assume that $J\colon V \to \mathbb{R}$ is Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. Suppose that J has a local minimum at $u \in V$; so that,
$$J(u) \leqslant J(w), \qquad \text{for all } w \in V \text{ with } \|w-u\| < \delta,$$
and some $\delta > 0$. Show that
$$dJ(u;v) = 0, \qquad \text{for all } v \in V_o$$