Assignment #8

Due on Friday, November 8, 2019

Read Section 5.4, *The Isoperimetric Theorem,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

- **The Isoperimetric Theorem**. For any simple, closed curve C in the xy–plane of perimeter ℓ and which encloses a region of area A, the following inequality holds true:
$$4\pi A \leqslant \ell^2. \tag{1}$$
Furthermore, equality in (1) is attained if and only if C is a circle.

- The inequality in (1) is known as the **isoperimetric inequality**.

- **Wirtinger's Inequality**. Let $f \in C_o^1([0,\pi],\mathbb{R})$ be such that
$$\lim_{t\to 0^+} f'(t) \quad \text{and} \quad \lim_{t\to \pi^-} f'(t) \quad \text{exist.}$$
Then,
$$\int_0^\pi (f(t))^2\,dt \leqslant \int_0^\pi (f'(t))^2\,dt. \tag{2}$$
Furthermore, equality in (2) is attained if and only if $f(t) = c\sin t$, for some constant c.

Do the following problems

1. Let $\mathcal{A}$ denote that class of simple, closed curves in the plane whose inside has area A. Use the isoperimetric theorem to find the shape of the curve in $\mathcal{A}$ that has the smallest possible perimeter. Explain the reasoning leading to your solution.

2. For given $b > 0$, put $V = C^1([0,b],\mathbb{R})$ and $V_o = C_o^1([0,b],\mathbb{R})$. Define functionals $J\colon V \to \mathbb{R}$ and $K\colon V \to \mathbb{R}$ by
$$J(y) = \int_0^b \sqrt{1+(y'(x))^2}\,dx, \qquad \text{for all } y \in V,$$

and

$$K(y) = \int_0^b y(x)\ dx, \qquad \text{for all } y \in V,$$

respectively. Let

$$\mathcal{A} = \{y \in V_o \mid y(x) \geqslant 0\}.$$

Use the isoperimetric theorem to solve the following constrained optimization problem:

Minimize $J(y)$ for $y \in \mathcal{A}$ subject to the constraint

$$K(y) = a,$$

for some $a > 0$.

3. Let $f \in C_o^1([0, \pi], \mathbb{R})$ be such that

$$\lim_{t\to 0^+} f'(x) \ \text{ and } \ \lim_{t\to \pi^-} f'(x) \ \text{ exist.} \tag{3}$$

(a) Use the assumption in (3) and L'Hospital's rule to deduce that

$$\lim_{t\to 0^+} \frac{\cos t}{\sin t} f(t) \ \text{ and } \ \lim_{t\to \pi^-} \frac{\cos t}{\sin t} f(t) \ \text{ exist.} \tag{4}$$

(b) Use the result in (4) to compute

$$\lim_{t\to 0^+} \frac{(\cos t)(f(t))^2}{\sin t} \ \text{ and } \ \lim_{t\to \pi^-} \frac{(\cos t)(f(t))^2}{\sin t}. \tag{5}$$

(c) Use integration by parts and the result from part (b) above to show that

$$\int_0^\pi 2\frac{\cos t}{\sin t} f(t) f'(t)\ dt = \int_0^\pi \frac{(f(t))^2}{\sin^2 t}\ dt. \tag{6}$$

Explain why both integrals in (6) are finite.

4. Let $f \in C_o^1([0, \pi], \mathbb{R})$ be such that the assumption in (3) is satisfied.

Expand the integrand in $\displaystyle\int_0^\pi \left[f'(t) - \frac{\cos t}{\sin t} f(t)\right]^2\ dt$ and use the result in (6) to derive the identity

$$\int_0^\pi \left[f'(x) - \frac{\cos t}{\sin t} f(t)\right]^2\ dt = \int_0^\pi (f'(t))^2\ dt - \int_0^\pi (f(t))^2\ dt. \tag{7}$$

5. **Proof of Wirtinger's Inequality**. Let $f \in C_o^1([0,\pi],\mathbb{R})$ be such that the assumption in (3) is satisfied.

   (a) Use the identity in (7) to deduce Wirtinger's inequality in (2).

   (b) Use the identity in (7) and the basic lemma I to deduce that, if equality holds true in (2), then f solves the ODE

   $$\frac{dy}{dt} - \frac{\cos t}{\sin t}\, y = 0, \qquad \text{for } 0 < t < \pi. \tag{8}$$

   (c) Use separation of variables to solve the ODE in (8) and deduce that equality in (2) holds true if and only if $y = c\sin t$, for $t \in [0,\pi]$, and for some constant c.