Assignment #10

Due on Wednesday, November 6, 2019

Read Section 4.4 on *Differentiable Paths* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.5.1 on *Differentiability of Paths* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- *(Parametrization)* Let I denote and interval of real numbers, $\sigma\colon I \to \mathbb{R}^n$ be a continuous path, and let C denote the image of I under σ. Then, C is called a curve in $\mathbb{R}^n$. If σ is one–to–one on I, then σ is called a parametrization of C. For example, if v and u are distinct vectors in $\mathbb{R}^n$, then

  $$\sigma(t) = u + t(v - u), \quad \text{for } 0 \leqslant t \leqslant 1,$$

  is a parametrization of the straight line segment from the point u to the point v in $\mathbb{R}^n$.

- *(C^1 Curves)* If C is parametrized by a C^1 path, $\sigma\colon I \to \mathbb{R}^n$, with $\sigma'(t) \neq \mathbf{0}$ for all $t \in I$, the curve C is said to be a C^1 curve, or a smooth curve.

- *(Simple Closed Curves)* If $\sigma\colon [a, b] \to \mathbb{R}^n$ is a parametrization of a curve C, with $\sigma(a) = \sigma(b)$ and $\sigma\colon [a, b) \to \mathbb{R}^n$ being one–to–one, then C is said to be a simple closed curve.

- *(The Jordan Curve Theorem)* Any simple closed curve, C, in the xy–plane divides the plane into two disjoint, connected open sets: a bounded region and an unbounded region. The bounded region is called the interior of the curve C, and the unbounded region is called the exterior of C.

Do the following problems

1. Give a C^1 parametrization of the ellipse $x^2 + 4y^2 = 1$. Find the points on the ellipse at which the tangent vector is parallel to the line $y = x$.

2. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the path defined by

   $$\sigma(t) = (e^{kt} \cos t, e^{kt} \sin t), \quad \text{for all } t \in \mathbb{R},$$

   where $k \neq 0$.

(a) Let $r(t) = \|\sigma(t)\|$ for all $t \in \mathbb{R}$ and explain why the image of σ is a spiral.

(b) Compute a unit vector which is tangent to the curve parametrized by σ at the point $\sigma(t)$ for all $t \in \mathbb{R}$.

(c) Compute the cosine of the angle between the tangent to the curve at $\sigma(t)$ and the vector connecting the origin in $\mathbb{R}^2$ to the point $\sigma(t)$. What do you conclude?

3. Let $\sigma\colon (a,b) \to \mathbb{R}^n$ and $\gamma\colon (a,b) \to \mathbb{R}^n$ be two differentiable paths defined on a common interval (a,b). Define the real valued function, $f\colon (a,b) \to \mathbb{R}$, by

$$f(t) = \sigma(t) \cdot \gamma(t), \qquad \text{for all } t \in (a,b).$$

(a) Show that f is differentiable on (a,b) and provide a formula for computing $f'(t)$ in terms of the $\sigma(t)$, $\gamma(t)$, and their corresponding tangent vectors.

(b) Suppose that $\sigma\colon (a,b) \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in (a,b)$. Use the result of the previous part to show that the function $r\colon (a,b) \to \mathbb{R}$ defined by $r(t) = \|\sigma(t)\|$ for all $t \in (a,b)$ is differentiable on (a,b) and compute its derivative.
Suggestion: Write $r(t) = \sqrt{\sigma(t) \cdot \sigma(t)}$ for all $t \in (a,b)$.

4. Let I denote an open interval and $\sigma\colon I \to \mathbb{R}^n$ denote a differentiable path with $\|\sigma(t)\| = c$, a positive constant, for all $t \in I$. Prove that the tangent vector, $\sigma'(t)$, to the curve at $\sigma(t)$ is orthogonal to $\sigma(t)$.

   Suggestion: Start with $\|\sigma(t)\|^2 = c^2$, or $\sigma(t) \cdot \sigma(t) = c^2$, for all $t \in I$.

5. Let $\sigma(t) = (x(t), y(t))$, for $t \in [a,b]$, be a parametrization of a simple closed curve. Assume that σ is oriented in the counterclockwise sense. Give the unit vector to the curve at $\sigma(t)$, for $t \in (a,b)$, which is perpendicular to $\sigma'(t)$ and points towards the exterior of the curve.