

Assignment #12

Due on Friday, November 15, 2019

Read Section 4.6 on *Derivatives of Composition* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Do the following problems

- Let $U = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ and define $f: U \rightarrow \mathbb{R}$ by $f(x, y) = \arctan\left(\frac{y}{x}\right)$, for all $(x, y) \in U$.
 - Compute the gradient of f in U .
 - Let I be an open interval and $\sigma: I \rightarrow U$ be a differentiable path given by $\sigma(t) = (x(t), y(t))$ for $t \in I$. Define $\theta: I \rightarrow \mathbb{R}$ by $\theta(t) = (f \circ \sigma)(t)$ for all $t \in I$. Apply the chain rule to verify that $\theta' = \frac{-yx' + xy'}{x^2 + y^2}$.
 - Apply the result from part (b) to the path $\sigma: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}^2$ given by $\sigma(t) = (\cos t, \sin t)$, for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- Suppose that the temperature in a region of space is given by a function $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$T(x, y, z) = kx^2(y - z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

and some positive constant k .

An insect flies in the region along a path modeled by a C^1 function $\sigma: \mathbb{R} \rightarrow \mathbb{R}^3$. Suppose that at time $t = 0$ the insect is located at $(0, 0, 0)$ and its velocity is $\sigma'(0) = \hat{i} + \hat{j} + 2\hat{k}$. Compute the rate of change of temperature sensed by the insect at time $t = 0$.

- Let I be an open interval of real numbers and U be an open subset of \mathbb{R}^n . Suppose that $\sigma: I \rightarrow \mathbb{R}^n$ is a differentiable path and that $f: U \rightarrow \mathbb{R}$ is a differentiable scalar field. Assume also that the image of I under σ , $\sigma(I)$, is contained in U . Suppose also that the derivative of the path σ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all } t \in I.$$

Show that if the gradient of f along the path σ is never zero, then f decreases along the path as t increases.

Suggestion: Use the chain rule to compute the derivative of $f(\sigma(t))$.

4. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors x and y in U , there exists a differentiable path $\sigma: [0, 1] \rightarrow \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0, 1]$; i.e., any two elements in U can be connected by a differentiable path whose image is entirely contained in U .

Suppose that U is an open, path connected subset of \mathbb{R}^n . Let $f: U \rightarrow \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that f must be constant.

5. Let U be an open subset of \mathbb{R}^n and $f: U \rightarrow \mathbb{R}$ be a differentiable scalar field defined on U . The function f is said to be homogeneous of order k if

$$f(tv) = t^k f(v),$$

for all $v \in U$ and all positive $t \in \mathbb{R}$ such that $tv \in U$.

- (a) Show that the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = xy$, for all $(x, y) \in \mathbb{R}^2$, is homogeneous of order 2.
- (b) Give examples of a scalar field which is homogeneous of order 1 and of a scalar field which is homogeneous of order 0.
- (c) Prove Euler's Theorem: If $f: U \rightarrow \mathbb{R}$ is differentiable and homogeneous of order k , then

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = kf(x),$$

for all $x = (x_1, x_2, \dots, x_n) \in U$.

Suggestion: Let $\sigma: \mathbb{R} \rightarrow \mathbb{R}^n$ be given by $\sigma(t) = tx$, for all $t \in \mathbb{R}$ and $x = (x_1, x_2, \dots, x_n) \in U$, and apply the chain rule to the composition $f \circ \sigma$.