Assignment #13

Due on Wednesday, November 20, 2019

Read Section 4.6 on *Derivatives of Compositions* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems

1. Let x and y be functions of u and v: $x = x(u, v)$, $y = y(u, v)$, and let $f(x, y)$ denote a scalar field. Find $\partial f/\partial u$ and $\partial f/\partial v$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

2. For f, x and y as in Problem 1, express $\dfrac{\partial^2 f}{\partial u^2}$ in terms of the partial derivatives of f with respect to x and y and the partial derivatives of x and y with respect to u. Assume that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

3. Let $G\colon \mathbb{R}^n \to \mathbb{R}^m$ and $F\colon \mathbb{R}^m \to \mathbb{R}^n$ be differentiable functions such that

$$(F \circ G)(x) = x, \qquad \text{for all } x \in \mathbb{R}^n.$$

Put $y = G(x)$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Apply the chain rule to show that

$$\frac{\partial f_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial f_i}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial f_i}{\partial y_m}\frac{\partial y_m}{\partial x_j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases}$$

where $f_1, f_2, \ldots, f_n\colon \mathbb{R}^m \to \mathbb{R}$ are the components of the vector field F.

4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2 + xy$, for all $(x, y) \in \mathbb{R}^2$, and assume that $x = r\cos\theta$ and $y = r\sin\theta$ for $r \geqslant 0$ and $\theta \in \mathbb{R}$. Put $z = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Use the chain rule to compute $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$

5. Let f be a scalar field defined for $(x, y) \in \mathbb{R}^2$, where $x = r\cos\theta$, $y = r\sin\theta$. Show that

$$\nabla f = \frac{\partial f}{\partial r}\widehat{\mathbf{u}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\mathbf{u}}_\theta,$$

where $\widehat{\mathbf{u}}_r = (\cos\theta, \sin\theta)$ and $\widehat{\mathbf{u}}_\theta = (-\sin\theta, \cos\theta)$.

Suggestion: First find $\partial f/\partial r$ and $\partial f/\partial \theta$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$ and then solve for $\partial f/\partial x$ and $\partial f/\partial y$ int terms of $\partial f/\partial r$ and $\partial f/\partial \theta$.