Assignment #14

Due on Friday, November 22, 2019

Read Section 5.1 on *Path Integrals* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- Let U be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be a continuous scalar field. Let $C \subset U$ be a C^1 simple curve. We define the integral of f over C, denoted $\displaystyle\int_C f\underset{\cdot}{s}$, to be

$$\int_C f\ ds = \int_a^b f(\sigma(t))\|\sigma'(t)\|\ dt,$$

where $\sigma\colon [a,b] \to \mathbb{R}^n$ is any C^1 parametrization of C.

- A curve, C, is said to be piece–wise C^1 if C can be decomposed into a finite union of C^1 simple curves, $C_1, C_2, \ldots, C_k$:

$$C = \bigcup_{i=1}^{k} C_i.$$

If $C \subset U$, where U is an open subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is a continuous scalar field, we define the integral of f over C by

$$\int_C f\ ds = \sum_{i=i}^{k} \int_{C_i} f\ ds.$$

Do the following problems

1. Consider a portion of a helix, C, parametrized by the path

$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$

Let $f(x,y,z) = x^2 + y^2 + z^2$ for all $(x,y,z) \in \mathbb{R}^3$. Evaluate

$$\int_C f\ ds$$

2. Find the mass of a wire that is parametrized by

$$C = \left\{ \left(\frac{3}{2}t^2, (1+2t)^{3/2} \right) \ \Big| \ 0 \leqslant t \leqslant 2 \right\}$$

and has a linear density (mass per unit length) given by $\rho(x, y) = 2x + 1$.

3. Let $f(x, y) = y$ for all $(x, y) \in \mathbb{R}^2$. For each of the following curves, C, in the xy–plane, evaluate $\int_C f \, ds$.

   (a) C is the segment along the x axis from $(0, 0)$ to $(1, 0)$.

   (b) C is the segment along the y axis from $(0, 0)$ to $(0, 1)$.

   (c) C is the unit circle in $\mathbb{R}^2$.

4. Evaluate $\int_C (x^3 - yz) \, ds$, where C is the intersection of the planes $x + y - z = 1$ and $z = 3x$ from $x = 0$ to $x = 1$.

5. Let C denote the boundary of the square

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}.$$

Evaluate the integral of $f(x, y) = xy^2$, for $(x, y) \in \mathbb{R}^2$, over C.

Note: Observe that C is not a C^1 curve, but it can be decomposed into an union of four simple, C^1 curves.