Assignment #7

Due on Monday, October 14, 2019

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes at
`http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- *(Continuous Functions 2)* Let U denote an open subset of $\mathbb{R}^n$. A function
  $F\colon U \to \mathbb{R}^m$ is continuous on U if and only if, for every open subset V of $\mathbb{R}^m$,
  the pre–image of V under F, denoted $F^{-1}(V)$, is open in $\mathbb{R}^n$.

- *(Composition of Continuous Functions)* Let U denote an open subset of $\mathbb{R}^n$ and
  Q an open subset of $\mathbb{R}^m$. Suppose that the maps $F\colon U \to \mathbb{R}^m$ and $G\colon Q \to \mathbb{R}^k$
  are continuous on their respective domains and that $F(U) \subseteq Q$. Then, the
  composition $G \circ F\colon U \to \mathbb{R}^k$ is continuous on U.

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is a scalar field
   and $G\colon U \to \mathbb{R}^m$ is vector–valued function.

   (a) Explain how the product fG is defined.

   (b) Prove that if both f and G are continuous on U, then the vector valued
   function fG is also continuous on U.

2. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be two scalar
   fields on U, and define $h\colon U \to \mathbb{R}$ by

   $$h(x, y) = f(x, y)g(x, y) \quad \text{for all } (x, y) \in U.$$

   Prove that if both f and g are continuous on U, then so is h.

   Suggestion: First prove that the function $G\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $G(x, y) = xy$
   for all $(x, y) \in \mathbb{R}^2$, is continuous. Then, let $F\colon U \to \mathbb{R}^2$ denote the map given
   by
   $$F(x, y) = (f(x, y), g(x, y)) \quad \text{for all } (x, y) \in U,$$
   and observe that $h = G \circ F$.

3. Let $U = \mathbb{R}^n \backslash \{\mathbf{0}\} = \{v \in \mathbb{R}^n \mid v \neq \mathbf{0}\}$.

   (a) Prove that U is an open subset of $\mathbb{R}^n$.

   (b) Define $f \colon U \to \mathbb{R}$ by

   $$f(v) = \frac{1}{\|v\|} \quad \text{for all } \ v \in U.$$

   Prove that f is continuous on U.

   Suggestion: Note that the function, g, defined by

   $$g(t) = \frac{1}{t} \quad \text{for all } t \neq 0,$$

   is continuous for $t \neq 0$.

4. Let $I \subseteq \mathbb{R}$ be an open interval and $\sigma \colon I \to \mathbb{R}^n$ be continuous path in $\mathbb{R}^n$ satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Define the function $f \colon I \to \mathbb{R}$ by

   $$f(t) = \frac{1}{\|\sigma(t)\|} \quad \text{for all } \ t \in I.$$

   Prove that f is continuous on I.

5. Let

   $$f(x,y) = \frac{x-y}{x+y}, \quad x + y \neq 0.$$

   Can this function be defined on the line $x + y = 0$ so that it is continuous at some point on this line?