Assignment #8

Due on Friday, October 25, 2019

Read Section 4.1 on *Definition of Differentiability* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.2 on *The Derivative* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.3 on *Differentiable Scalar Fields* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems

1. Let f denote a real valued function defined on some open interval around $a \in \mathbb{R}$. Consider a line of slope m and equation

$$L(x) = f(a) + m(x-a) \quad \text{for all} \quad x \in \mathbb{R}.$$

Suppose that this line is the best approximation to the function f at a in the sense that

$$\lim_{x \to a} \frac{|E(x)|}{|x-a|} = 0,$$

where $E(x) = f(x) - L(x)$ for all x in the interval in which f is defined.

Prove that f is differentiable at a, and that $f'(a) = m$.

2. Let U be an open subset for $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be differentiable at $u \in U$ if and only if there exists a unique linear transformation $T_u\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\|v-u\| \to 0} \frac{\|F(v) - F(u) - T_u(v-u)\|}{\|v-u\|} = 0.$$

Prove that if F is differentiable at u, then it is also continuous at u.

Give an example that shows that the converse of this assertion is not true.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0,0)$.

4. Is $f(x,y,z) = x\sqrt{y^2+z^2}$ differentiable at $(0,0,0)$? Prove your assertion.

5. Is the scalar field

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

continuous at the origin? Is it differentiable at the origin?