Solutions to Review Problems for Exam 2

1. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

   Solution: Let u and w be any vector in $\mathbb{R}^n$ and consider

$$\begin{aligned} f(u+w) &= \frac{1}{2}\|u+w\|^2 \\ &= \frac{1}{2}(u+w)\cdot(u+w) \\ &= \frac{1}{2}u\cdot u + u\cdot w + \frac{1}{2}w\cdot w \\ &= \frac{1}{2}\|u\|^2 + u\cdot w + \frac{1}{2}\|w\|^2; \end{aligned}$$

   so that,

$$f(u+w) = f(u) + u\cdot w + \frac{1}{2}\|w\|^2. \tag{1}$$

   The equation in (1) suggests that we set

$$Df(u)w = u\cdot w, \qquad \text{for } u, w \in \mathbb{R}^n, \tag{2}$$

   and

$$E(u;w) = \frac{1}{2}\|w\|^2, \qquad \text{for } u, w \in \mathbb{R}^n. \tag{3}$$

   Note that

$$\frac{E(u;w)}{\|w\|} = \frac{1}{2}\|w\|, \qquad \text{for } w \neq \mathbf{0}.$$

   Consequently,

$$\lim_{\|w\|\to 0} \frac{|E(u;w)|}{\|w\|} = 0.$$

   Thus, in view of (1), (3) and (3), we have shown that f is differentiable at u with derivative map $Df(u)$ given in (2). We therefore see that $\nabla f(u) = u$ for all $u \in \mathbb{R}^n$. □

   Alternate Solution: Alternatively, writing $(x_1, x_2, \ldots, x_n)$ for u, we have that

$$f(u) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2), \qquad \text{for all } u \in \mathbb{R}^n.$$

Then, the partial derivatives of f are

$$\frac{\partial f}{\partial x_i} = x_1, \qquad \text{for } i = 1, 2, \ldots, n, \tag{4}$$

which are continuous functions in $\mathbb{R}^n$. Thus, f is C^1 map and is therefore differentiable.

According to (4), the gradient of f is given by

$$\nabla f(u) = (x_1, x_2, \ldots, x_n) = u, \qquad \text{for all } u \in \mathbb{R}^n.$$

$\square$

2. Let $g\colon [0, \infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x, y) = g(r)$ where $r = \sqrt{x^2 + y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

   ***Solution**:* Take the partial derivative of $r^2 = x^2 + y^2$ on both sides with respect to x to obtain

   $$\frac{\partial (r^2)}{\partial x} = 2x.$$

   Applying the chain rule on the left–hand side we get

   $$2r\frac{\partial r}{\partial x} = 2x,$$

   which leads to

   $$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

   Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$. $\square$

   (b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

   ***Solution**:* Take the partial derivative of $f(x, y) = g(r)$ on both sides with respect to x and apply the chain rule to obtain

   $$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}.$$

   Similarly, $\dfrac{\partial f}{\partial y} = g'(r)\dfrac{y}{r}$.

It then follows that

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\widehat{i} + \frac{\partial f}{\partial y}\widehat{j}\\ &= g'(r)\frac{x}{r}\widehat{i} + g'(r)\frac{y}{r}\widehat{j}\\ &= \frac{g'(r)}{r}(x\widehat{i} + y\widehat{j})\\ &= \frac{g'(r)}{r}\mathbf{r}.\end{aligned}$$

□

3. Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset U of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0}\frac{f(v+t\widehat{u}) - f(v)}{t}$$

exists, we call it the *directional derivative of f at v in the direction of the unit vector $\widehat{u}$*. We denote it by $D_{\widehat{u}}f(v)$.

(a) Show that if f is differentiable at $v \in U$, then, for any unit vector $\widehat{u}$ in $\mathbb{R}^n$, the directional derivative of f in the direction of $\widehat{u}$ at v exists, and

$$D_{\widehat{u}}f(v) = \nabla f(v)\cdot\widehat{u},$$

where $\nabla f(v)$ is the gradient of f at v.

Proof: Suppose that f is differentiable at $v \in U$. Then,

$$f(v+w) = f(v) + Df(v)w + E(w),$$

where

$$Df(v)w = \nabla f(v)\cdot w,$$

and

$$\lim_{\|w\|\to 0}\frac{|E(w)|}{\|w\|} = 0.$$

Thus, for any $t \in \mathbb{R}$,

$$f(v+t\widehat{u}) = f(v) + t\nabla f(v)\cdot\widehat{u} + E(t\widehat{u}),$$

where

$$\lim_{|t|\to 0} \frac{|E(t\widehat{u})|}{|t|} = 0,$$

since $\|t\widehat{u}\| = |t|\|\widehat{u}\| = |t|$.
We then have that, for $t \neq 0$,

$$\frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} = \frac{E(t\widehat{u})}{t},$$

and consequently

$$\left| \frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} \right| = \frac{|E(t\widehat{u})|}{|t|},$$

from which we get that

$$\lim_{t\to 0} \left| \frac{f(v + t\widehat{u}) - f(v)}{t} - \nabla f(v) \cdot \widehat{u} \right| = 0.$$

This proves that

$$\lim_{t\to 0} \frac{f(v + t\widehat{u}) - f(v)}{t} = \nabla f(v) \cdot \widehat{u};$$

so that, the directional derivative of f in the direction of $\widehat{u}$ at v exists, and $D_{\widehat{u}} f(v) = \nabla f(v) \cdot \widehat{u}$. □

(b) Suppose that $f : U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\widehat{u}} f(v) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

Proof: Suppose, by way of contradiction, that $\nabla f(v) \neq \mathbf{0}$, and put

$$\widehat{u} = \frac{1}{\|\nabla f(v)\|} \nabla f(v).$$

Then, $\widehat{u}$ is a unit vector, and therefore, by the assumption,

$$D_{\widehat{u}} f(v) = 0,$$

or

$$\nabla f(v) \cdot \widehat{u} = 0.$$

But this implies that

$$\nabla f(v) \cdot \frac{1}{\|\nabla f(v)\|} \nabla f(v) = 0,$$

where

$$\begin{aligned}\nabla f(v)\cdot\frac{1}{\|\nabla f(v)\|}\nabla f(v) &= \frac{1}{\|\nabla f(v)\|}\nabla f(v)\cdot\nabla f(v)\\ &= \frac{1}{\|\nabla f(v)\|}\|\nabla f(v)\|^2\\ &= \|\nabla f(v)\|.\end{aligned}$$

It then follows that $\|\nabla f(v)\| = 0$, which contradicts the assumption that $\nabla f(v) \neq \mathbf{0}$. Therefore, $\nabla f(v)$ must be the zero vector. □

(c) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}}f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(v)$.

Proof. If f is differentiable at x, then $D_{\widehat{u}}f(x) = \nabla f(x)\cdot\widehat{u}$, as was shown in part (a). Thus, by the Cauchy–Schwarz inequality,

$$|D_{\widehat{u}}f(x)| \leqslant \|\nabla f(x)\|\|\widehat{u}\| = \|\nabla f(x)\|,$$

since $\widehat{u}$ is a unit vector. Hence,

$$-\|\nabla f(x)\| \leqslant D_{\widehat{u}}f(x) \leqslant \|\nabla f(x)\|$$

for any unit vector $\widehat{u}$, and so the largest value that $D_{\widehat{u}}f(x)$ can have is $\|\nabla f(x)\|$.

If $\nabla f(x) \neq \mathbf{0}$, then $\widehat{u} = \dfrac{1}{\|\nabla f(x)\|}\nabla f(x)$ is a unit vector, and

$$\begin{aligned}D_{\widehat{u}}f(x) &= \nabla f(x)\cdot\widehat{u}\\ &= \nabla f(x)\cdot\frac{1}{\|\nabla f(x)\|}\nabla f(x)\\ &= \frac{1}{\|\nabla f(x)\|}\nabla f(x)\cdot\nabla f(x)\\ &= \frac{1}{\|\nabla f(x)\|}\|\nabla f(x)\|^2\\ &= \|\nabla f(x)\|.\end{aligned}$$

Thus, $D_{\widehat{u}}f(x)$ attains its largest value when $\widehat{u}$ is in the direction of $\nabla f(x)$.

□

4. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $u \in U$. Fix u and v in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(u + t(v-u)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function g is well defined.

   Answer: Since U is convex, for any $u, v \in U$, $u + t(v-u) \in U$ for all $t \in [0,1]$. Thus, $f(u + t(v-u))$ is defined for all $t \in [0,1]$, because f is defined on U. □

   (b) Show that g is differentiable on $(0,1)$ and that

$$g'(t) = \nabla f(u + t(v-u)) \cdot (v-u) \quad \text{for} \quad 0 < t < 1.$$

   Solution: It follows from the chain rule that the composition $g = f \circ \sigma\colon [0,1] \to \mathbb{R}$, where $\sigma\colon [0,1] \to \mathbb{R}^n$ is the path given by

$$\sigma(t) = u + t(v-u), \qquad \text{for all } t \in [0,1],$$

   is differentiable in $(0,1)$ and

$$g'(t) = \nabla f(\sigma(t)) \cdot \sigma'(t), \qquad \text{for all } t \in (0,1),$$

   where

$$\sigma(t) = v - u, \qquad \text{for all } t.$$

   Consequently, we get that

$$g'(t) = \nabla f(u + t(v-u)) \cdot (v-u) \quad \text{for} \quad 0 < t < 1.$$

   □

   (c) Use the mean value theorem for derivatives to show that there exists a point z is the line segment connecting u to v such that

$$f(v) - f(u) = D_{\widehat{w}} f(z) \|v - u\|, \tag{5}$$

   where $\widehat{w}$ is the unit vector in the direction of the vector $v - u$; that is, $\widehat{w} = \dfrac{1}{\|v-u\|}(v-u)$.

   Solution: The mean value theorem implies that there exists $\tau \in (0,1)$ such that

$$g(1) - g(0) = g'(\tau)(1-0),$$

so that

$$f(v) - f(u) = \nabla f(u + \tau(v-u)) \cdot (v-u). \tag{6}$$

Put $z = u + \tau(v-u)$ and $\widehat{w} = \dfrac{1}{\|v-u\|}(v-u)$. We can then write (6) as

$$\begin{aligned} f(v) - f(u) &= \left(\nabla f(z) \cdot \frac{1}{\|v-u\|}(v-u)\right) \|v-u\| \\ &= (\nabla f(z) \cdot \widehat{w})\, \|v-u\|, \end{aligned}$$

which yields (5). □

(d) Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

Solution: Fix $u_o \in U$. Then, for any $u \in U$, the formula in (5) yields

$$f(u) - f(u_o) = D_{\widehat{w}} f(z) \|u - u_o\|, \tag{7}$$

where $D_{\widehat{w}} f(z) = \nabla f(z) \cdot \widehat{w} = 0$ by the assumption. Hence, it follows from (7) that

$$f(u) = f(u_o), \quad \text{for all } u \in U;$$

in other words, f is constant in U. □

5. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

Solution: If f has a local maximum or minimum at $\sigma(t_o)$, then $g'(t_o) = 0$, where, by the chain rule,

$$g'(t) = \nabla f(\sigma(t)) \cdot \sigma'(t) \quad \text{for all} \quad t \in I.$$

It then follows that

$$\nabla f(\sigma(t_o)) \cdot \sigma'(t_o) = 0,$$

and, consequently, $\nabla f(\sigma(t_o)$ is perpendicular to the tangent to the path at $\sigma(t_o)$. □

6. Let C denote the boundary of the oriented triangle, $T = [(0,0)(1,0)(1,2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\displaystyle\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x$, by applying the Fundamental Theorem of Calculus.

   Solution*:* Apply the Fundamental Theorem of Calculus to the 1–form

$$\omega = -\frac{y^2}{2}\ \mathrm{d}x + \frac{x^2}{2}\ \ \mathrm{d}y$$

   over the oriented triangle T; namely,

$$\int_{\partial T} \omega = \int_T d\omega,$$

   where

$$d\omega = (x+y)\ dx \wedge dy.$$

   Thus, since T is positively oriented, it follows that

$$\begin{aligned}\int_{\partial T} \omega &= \iint_T (x+y)\ dxdy \\ &= \int_0^1 \int_0^{2x} (x+y)\ dydx \\ &= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{2x} dx \\ &= \int_0^1 4x^2 dx,\end{aligned}$$

   so that

$$\int_C \frac{x^2}{2}\ \ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x = \frac{4}{3}.$$

□

7. Let $F(x,y) = 2x\ \widehat{\imath} - y\ \widehat{\jmath}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

   Solution*:* Apply the Fundamental Theorem of Calculus,

$$\oint_{\partial R} F \cdot \widehat{n}\ ds = \int_R d\omega,$$

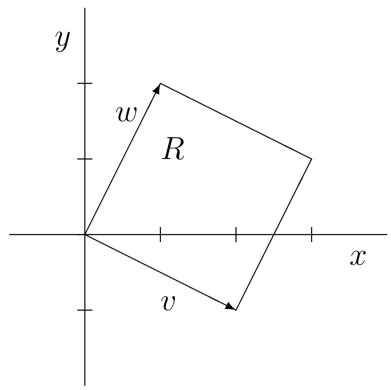

Figure 1: Sketch of Region R in Problem 7

where

$$\omega = P\ dy - Q\ dx = 2x\ dy - (-y)\ dx = y\ dx + 2x\ dy,$$

so that

$$d\omega = dy \wedge dx + 2dx \wedge dy = dx \wedge dy,$$

we obtain that

$$\begin{aligned} \oint_{\partial R} F \cdot\ d\mathbf{n} &= \int_R dx \wedge dy \\ &= \iint_R dxdy \\ &= \text{area}(R). \end{aligned}$$

To find the area of the region R, shown in Figure 1, observe that R is a parallelogram determined by the vectors $v = 2\,\widehat{i} - \widehat{j}$ and $w = \widehat{i} + 2\,\widehat{j}$. Thus,

$$\text{area}(R) = \|v \times w\| = 5.$$

It the follows that

$$\oint_{\partial R} F \cdot n \ \ \text{d}s = \iint_R \text{d}x\ \text{d}y = 5.$$

□

8. Evaluate the line integral $\int_{\partial R} (x^4 + y) \ \ \text{d}x + (2x - y^4) \ \ \text{d}y$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

***Solution**:* Apply the Fundamental Theorem of Calculus to get

$$\begin{aligned}\int_{\partial R} (x^4 + y)\ dx + (2x - y^4)\ dy &= \int_R d(x^4 + y) \wedge dx + d(2x - y^4) \wedge dy \\ &= \int_R dy \wedge dx + 2dx \wedge dy \\ &= \int_R dx \wedge dy \\ &= \text{area}(R) \\ &= 12.\end{aligned}$$

□

9. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

***Solution**:* The region, R, is sketched in Figure 2. We evaluate the double

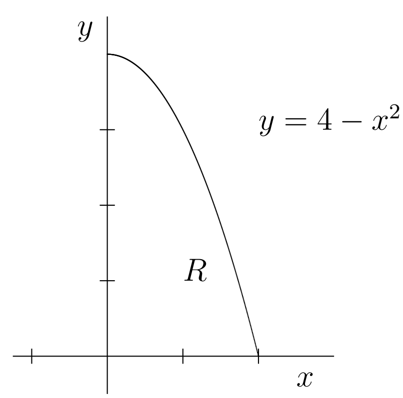

Figure 2: Sketch of Region R in Problem 11

integral, $\iint_R xy^2 \ \mathrm{d}x\ \mathrm{d}y$, as an iterated integral

$$\begin{aligned}\iint_R xy^2 \ \mathrm{d}x\ \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^2 \int_0^{4-x^2} xy^2 \ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^2 \frac{xy^3}{3}\Big|_0^{4-x^2} \ \mathrm{d}x \\ &= \frac{1}{3}\int_0^2 x(4-x^2)^3 \ \mathrm{d}x.\end{aligned}$$

To evaluate the last integral, make the change of variables: $u = 4 - x^2$. We then have that $\mathrm{d}u = -2x\ \mathrm{d}x$ and

$$\begin{aligned}\iint_R xy^2 \ \mathrm{d}x\ \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \ \mathrm{d}y\ \mathrm{d}x \\ &= -\frac{1}{6}\int_4^0 u^3 \ \mathrm{d}u \\ &= \frac{1}{6}\int_0^4 u^3 \ \mathrm{d}u.\end{aligned}$$

Thus,

$$\iint_R xy^2 \ \mathrm{d}x\ \mathrm{d}y = \frac{4^4}{24} = \frac{32}{3}.$$

□

10. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{8}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x$, where ∂R is the ellipse in (8) traversed in the positive sense.

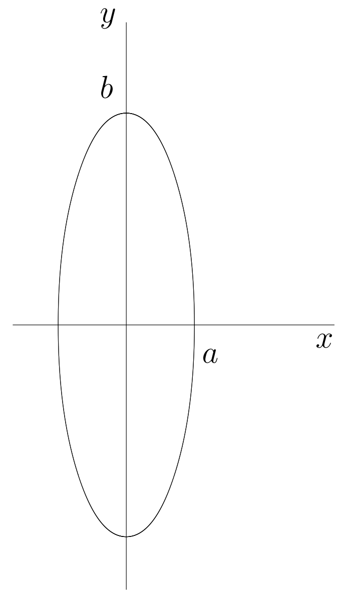

Figure 3: Sketch of ellipse

***Solution**:* A sketch of the ellipse is shown in Figure 3 for the case $a < b$. A parametrization of the ellipse is given by

$$x = a\cos t, \quad y = b\sin t, \quad \text{for } \ 0 \leqslant t \leqslant 2\pi.$$

We then have that $\mathrm{d}x = -a\sin t\ \mathrm{d}t$ and $\mathrm{d}y = b\cos t\ \mathrm{d}t$. Therefore

$$\begin{aligned}
\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x &= \int_0^{2\pi} [a\cos t \cdot b\cos t - b\sin t \cdot (-a\cos t)]\ \mathrm{d}t \\
&= \int_0^{2\pi} [ab\cos^2 t + ab\sin^2 t]\ \mathrm{d}t \\
&= ab\int_0^{2\pi} (\cos^2 t + ab\sin^2 t)\ \mathrm{d}t \\
&= ab\int_0^{2\pi}\ \mathrm{d}t \\
&= 2\pi ab.
\end{aligned}$$

□

(b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (8).

Solution: Let $F(x,y) = x\,\widehat{i} + y\,\widehat{j}$. Then,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \oint_{\partial R} F\cdot n\ \ \mathrm{d}s.$$

Thus, by Green's Theorem in divergence form,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\mathrm{div}F(x,y) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

Consequently,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = 2\iint_R\ \mathrm{d}x\ \mathrm{d}y = 2\ \mathrm{area}(R).$$

It then follows that

$$\mathrm{area}(R) = \frac{1}{2}\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x.$$

Thus,

$$\mathrm{area}(R) = \pi ab,$$

by the result in part (a). □

11. Evaluate the double integral $\displaystyle\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 4.

Solution: Compute

$$\begin{aligned}\iint_R e^{-x^2}\ \ \mathrm{d}x\ \mathrm{d}y &= \int_0^2\int_0^{2x} e^{-x^2}\ \ \mathrm{d}y\ \mathrm{d}x\\ &= \int_0^2 2xe^{-x^2}\ \ \mathrm{d}x\\ &= \left[-e^{-x^2}\right]_0^2\\ &= 1-e^{-4}.\end{aligned}$$

□

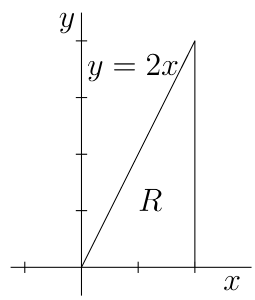

Figure 4: Sketch of Region R in Problem 11