Assignment #2

Due on Friday February 1, 2008

Read Section 1.4 on Set Theory, pp. 6–11, in DeGroot and Schervish.

Background and Definitions

• Recall that a $\sigma$-field, $\mathcal{B}$, is a collection of subsets of a sample space $\mathcal{C}$, referred to as events, which satisfy:

  (1) $\emptyset \in \mathcal{B}$ ($\emptyset$ denotes the empty set)
  (2) If $E \in \mathcal{B}$, then its complement, $E^c$, is also an element of $\mathcal{B}$.
  (3) If $\{E_1, E_2, E_3 \ldots\}$ is a sequence of events, then

\[
E_1 \cup E_2 \cup E_3 \cup \ldots = \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.
\]

• Let $\mathcal{S}$ denote a collection of subsets of a sample space $\mathcal{C}$. The $\sigma$–field generated by $\mathcal{S}$, denoted by $\mathcal{B}(\mathcal{S})$, is the smallest $\sigma$–field in $\mathcal{C}$ which contains $\mathcal{S}$.

• $\mathcal{B}_b$ denotes the Borel $\sigma$–field of the real line, $\mathbb{R}$. This is the $\sigma$–field generated by the semi–infinite intervals

\[(-\infty, b], \quad \text{for } b \in \mathbb{R}.\]

Do the following problems

1. Let $A$, $B$ and $C$ be subsets of a sample space $\mathcal{C}$. Prove the following

   (a) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
   (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

2. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a $\sigma$–field of subsets of $\mathcal{C}$. Prove that if $\{E_1, E_2, E_3 \ldots\}$ is a sequence of events in $\mathcal{B}$, then

\[
\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.
\]

Hint: Use De Morgan’s Laws.
3. Let $C$ be a sample space and $B$ be a $\sigma$–field of subsets of $C$. For fixed $B \in B$ define the collection of subsets

$$
B_B = \{ D \subset C \mid D = E \cap B \text{ for some } E \in B \}.
$$

Show that $B_B$ is a $\sigma$–field.

*Note:* In this case, the complement of $D \in B_B$ has to be understood as $B \setminus D$; that is, the complement relative to $B$. The $\sigma$–field $B_B$ is the $\sigma$–field $B$ restricted to $B$, or *conditioned on* $B$.

4. Let $S$ denote the collection of all bounded, open intervals $(a, b)$, where $a$ and $b$ are real numbers with $a < b$. Show that

$$
B(S) = B_o;
$$

that is, the $\sigma$–field generated by bounded open intervals is the Borel $\sigma$–field.

*Hints:*

- We have already seen in the lecture that $B_o$ contains all bounded open intervals.
- Observe also that the semi–infinite open interval $(b, \infty)$ can be expressed as the union of the sequence of bounded intervals $(b, k)$, for $k = 1, 2, 3, \ldots$

5. Show that for every real number $a$, the singleton $\{ a \}$ is in the Borel $\sigma$–field $B_o$.

*Hint:* Express $\{ a \}$ as an intersection of a sequence of open intervals.