Solutions to Assignment #14

1. [Exercise 2 on page 127 in the text]

Suppose that in an electric display sign there are three light bulbs in the first row and four light bulbs in the second row. Let $X$ denote the number of bulbs in the first row that will be burned out at a specified time $t$, and let $Y$ denote the number of bulbs in the second row that will be burned out at the same time $t$. Suppose that the joint pmf of $X$ and $Y$ is as specified in Table 1:

<table>
<thead>
<tr>
<th>$X \backslash Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.08</td>
<td>0.07</td>
<td>0.06</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>1</td>
<td>0.06</td>
<td>0.10</td>
<td>0.12</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.06</td>
<td>0.09</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 1: Joint Probability Distribution for $X$ and $Y$, $p_{(X,Y)}$

Determine each of the following probabilities:

(a) $\Pr(X = 2)$
(b) $\Pr(Y \geq 2)$
(c) $\Pr(X \leq 2 \text{ and } Y \leq 2)$
(d) $\Pr(X = Y)$
(e) $\Pr(X > Y)$

Solution:

(a) Add probabilities along the third row:

$$\Pr(X = 2) = 0.05 + 0.06 + 0.09 + 0.04 + 0.03 = 0.27.$$ 

(b) Compute

$$\Pr(Y \geq 2) = \sum_{j=2}^{4} \Pr(Y = j).$$

This is the sum of the probabilities on the last three columns; thus,

$$\Pr(Y \geq 2) = 0.53.$$
(c) Add entries on the first three rows and columns:

\[ \Pr(X \leq 2 \text{ and } Y \leq 2) = 0.69. \]

(d) Add entries along the “main diagonal:”

\[ \Pr(X = Y) = 0.08 + 0.10 + 0.09 + 0.03 = 0.30. \]

(e) Add entries below the “main diagonal:”

\[ \Pr(X > Y) = 0.06 + 0.05 + 0.02 + 0.06 + 0.03 + 0.03 = 0.23. \]

\[ \square \]

2. [Exercise 4 on page 127 in the text]

Suppose that \(X\) and \(Y\) have a continuous joint distribution for which the pdf is defined as follows:

\[ f(x, y) = \begin{cases} 
  cy^2 & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\
  0 & \text{otherwise.}
\end{cases} \]

Determine

(a) the value of the constant \(c\);
(b) \(\Pr(X + Y > 2)\);
(c) \(\Pr(Y < 1/2)\);
(d) \(\Pr(X \leq 1)\);
(e) \(\Pr(X = 3Y)\).

**Solution:**

(a) We find \(c\) so that \(\iint_{\mathbb{R}^2} f(x, y) \, dx \, dy = 1\); that is, so that

\[ c \int_0^1 \int_0^2 y^2 \, dx \, dy = 1, \]

or

\[ \frac{2}{3} c = 1, \]

from which we get that \(c = 3/2\).
(b) Compute
\[ \Pr(X + Y > 2) = \int\int_A f(x, y) \, dxdy, \]
where
\[ A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1, \, x + y > 2\}. \]
The region \( A \) is sketched in Figure 1:

\[ \text{Figure 1: Sketch of region } A \]

We then have that
\[ \Pr(X + Y > 2) = \int_0^1 \int_{2-y}^2 \frac{3}{2} y^2 \, dxdy \]
\[ = \int_0^1 \left[ \frac{3}{2} y^2 \right]_0^{2-y} \, dy \]
\[ = \int_0^1 \frac{3}{2} y^3 \, dy \]
\[ = \frac{3}{8}. \]

(c) Compute
\[ \Pr(Y < 1/2) = \int\int_A f(x, y) \, dxdy, \]
where
\[ A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1/2\}. \]
The region \( A \) is sketched in Figure 2:
We then have that
\[
\Pr(Y > 1/2) = \int_{0}^{1/2} \int_{0}^{2} \frac{3}{2} y^2 \, dx \, dy
\]
\[
= \int_{0}^{1/2} \frac{3}{2} y^2 \left| x \right|_0^2 \, dy
\]
\[
= \int_{0}^{1/2} 3y^2 \, dy
\]
\[
= y^3 \left|_0^{1/2} \right.
\]
\[
= \frac{1}{8}.
\]

(d) Compute
\[
\Pr(X \leq 1) = \iint_{A} f(x, y) \, dx \, dy,
\]
where
\[A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}.
\]
The region $A$ is sketched in Figure 3:
Figure 3: Sketch of region $A$

We then have that

$$
\Pr(Y > 1/2) = \int_0^1 \int_0^1 \frac{3}{2}y^2\,dx\,dy
= \int_0^1 \frac{3}{2}y^2\,x\bigg|_0^1 \,dy
= \int_0^1 \frac{3}{2}y^2\,dy
= \frac{1}{2}y^3\bigg|_0^1
= \frac{1}{2}.
$$

(e) $\Pr(X = 3Y) = 0$.

3. *Exercise 6 on page 127 in the text*

Suppose a point $X$ is chosen at random from a region $S$ in the $xy$–plane containing all points $(x, y)$ such that $x \geq 0$, $y \geq 0$, and $4y + x \leq 4$.

(a) Determine the joint pdf of $X$ and $Y$.

(b) Suppose that $S_o$ is a subset of the region $S$ having area $\alpha$, and determine $\Pr[(X, Y) \in S_o]$.
Solution:

(a) The region $S$ is sketched in Figure 4:
We want $(X,Y)$ to have uniform distribution on $S$. We therefore define the joint distribution pdf of $X$ and $Y$ to be:

$$f_{(X,Y)}(x,y) = \begin{cases} 
\frac{1}{2} & \text{if } (x,y) \in S, \\
0 & \text{otherwise}.
\end{cases}$$

(b) Compute

$$\Pr[(X,Y) \in S_o] = \iint_{S_o} f_{(X,Y)}(x,y) \, dx\, dy$$

$$= \iint_{S_o} \frac{1}{2} \, dx\, dy$$

$$= \frac{1}{2} \int_{S_o} dx\, dy$$

$$= \frac{1}{2} \int_{S_o} \int dy$$

$$= \frac{1}{2} \cdot \text{area}(S_o)$$

$$= \frac{\alpha}{2}.$$
4. [Exercise 2 on page 135 in the text]

Suppose that $X$ and $Y$ have a discrete distribution for which the joint pmf is defined as follows:

$$p_{(X,Y)}(x, y) = \begin{cases} 
\frac{1}{30}(x + y) & \text{for } x = 0, 1, 2 \text{ and } y = 0, 1, 2, 3, \\
0 & \text{otherwise.}
\end{cases}$$

(a) Determine the marginal pmfs of $X$ and $Y$.

(b) Are $X$ and $Y$ independent?

**Solution:**

(a) The marginal pmf of $X$ is

$$p_X(x) = \sum_{y=0}^{3} p_{(X,Y)}(x, y)$$

$$= \frac{1}{30} \sum_{y=0}^{3} (x + y)$$

$$= \frac{1}{30} \sum_{y=0}^{3} x + \frac{1}{30} \sum_{y=0}^{3} y$$

$$= \frac{4}{30}x + \frac{6}{30}$$

$$= \frac{2}{15}x + \frac{1}{5},$$

for $x = 0, 1, 2$. 
Similarly, the marginal pmf of $Y$ is

$$p_Y(y) = \sum_{x=0}^{2} p_{(X,Y)}(x, y)$$

$$= \frac{1}{30} \sum_{x=0}^{2} (x + y)$$

$$= \frac{1}{30} \sum_{x=0}^{2} x + \frac{1}{30} \sum_{y=0}^{2} y$$

$$= \frac{3}{30} + \frac{3}{30} y$$

$$= \frac{1}{10} + \frac{1}{10} y,$$

for $y = 0, 1, 2, 3$.

(b) Observe that

$$p_X(x) \cdot p_Y(y) = \frac{1}{150} (2x+3)(y+1), \text{ for } x = 0, 1, 2 \text{ and } y = 0, 1, 2, 3,$$

which is not the same as $p_{(X,Y)}(x, y)$. It then follows that $X$ and $Y$ are not independent.

\[\square\]

5. [Exercise 4 on page 136 in the text]

Suppose the joint pdf of $X$ and $Y$ is as follows:

$$f_{(X,Y)}(x, y) = \begin{cases} 
\frac{15}{4}x^2 & \text{for } 0 \leq y \leq 1 - x^2 \\
0 & \text{otherwise.} 
\end{cases}$$

(a) Determine the marginal pdfs of $X$ and $Y$.

(b) Are $X$ and $Y$ independent?

**Solution:**
(a) The marginal distribution of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y) \, dy = \int_{0}^{1-x^2} \frac{15}{4} x^2 \, dy,$$

for $-1 < x < 1$. (see Figure 5).

![Figure 5: Sketch of region](image)

Thus,

$$f_X(x) = \begin{cases} 
\frac{15}{4} x^2 (1 - x^2) & \text{if } -1 < x < 1, \\
0 & \text{otherwise.} 
\end{cases}$$

To find the marginal distribution of $Y$ consider Figure 6:

![Figure 6: Sketch of region](image)
\[ f_Y(y) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) \, dx \]

\[ = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} 15 \frac{x^2}{4} \, dx \]

\[ = 2 \int_{0}^{\sqrt{1-y}} 15 \frac{x^2}{4} \, dx \]

\[ = \frac{5}{2} (1 - y)^{3/2}, \]

for \( 0 < y < 1 \).
It then follows that

\[ f_Y(y) = \begin{cases} \frac{5}{2} (1 - y)^{3/2} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases} \]

(b) Observe that

\[ f_X(x) \cdot f_Y(y) = \frac{75}{8} x^2 (1 - x^2)(1 - y)^{3/2}, \]

which is not the same as the given joint pdf. Consequently, \( X \) and \( Y \) are not independent.