Solutions to Assignment #1

1. Let $\mathcal{C}$ denote a sample space and $A$ be a subset of $\mathcal{C}$. Establish the following set theoretic identities:

(a) $A \cap \emptyset = \emptyset$,
(b) $A \cup \emptyset = A$;

where $\emptyset$ denotes the empty set. Justify your steps.

**Solution:**

(a) **Proof:** The set of elements that are common to both $A$ and $\emptyset$ is the empty set since $\emptyset$ has no elements. It then follows that

$$A \cap \emptyset = \emptyset.$$ □

(b) **Proof:** $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$. However, $\emptyset$ has no elements. It then follows that $x \in A$. We then have that

$$A \cup \emptyset \subseteq A.$$ On the other hand,

$$A \subseteq A \cup \emptyset.$$ Hence, $A \cup \emptyset = A$. □

2. Let $\mathcal{C}$ denote a sample space and $A$ and $B$ denote subsets of $\mathcal{C}$. Establish the following set theoretic identities:

(a) $(A^c)^c = A$,
(b) $(A \cup B)^c = A^c \cap B^c$;

where $A^c$ denote the complement of $A$.

**Solution:**

(a) **Proof:**

$$x \in (A^c)^c \quad \text{iff} \quad x \notin A^c$$

iff it is not true that $x \in A^c$

iff it is the case that $x \in A$

iff $x \in A$

Hence, $(A^c)^c = A$. □
(b) \textit{Proof:}

\[ x \in (A \cup B)^c \quad \text{iff} \quad x \notin A \cup B \]
\[ \quad \text{iff} \quad x \notin A \text{ and } x \notin B \]
\[ \quad \text{iff} \quad x \in A^c \text{ and } x \in B^c \]
\[ \quad \text{iff} \quad x \in A^c \cap B^c \]

Hence, \((A \cup B)^c = A^c \cap B^c\). \qed

3. Let \(C\) denote a sample space and \(A\), \(B\) and \(C\) denote subsets of \(C\). Prove the following distributive properties:

(a) \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\)
(b) \(A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\)

\textbf{Solution:}

(a) \textit{Proof:}

\[ x \in A \cap (B \cup C) \quad \text{iff} \quad x \in A \text{ and } x \in B \cup C \]
\[ \quad \text{iff} \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \]
\[ \quad \text{iff} \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \]
\[ \quad \text{iff} \quad x \in A \cap B \text{ or } x \in A \cap C \]
\[ \quad \text{iff} \quad x \in (A \cap B) \cup (A \cap C). \]

Consequently, \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\). \qed

(b) \textit{Proof:}

\[ x \in A \cup (B \cap C) \quad \text{iff} \quad x \in A \text{ or } x \in B \cap C \]
\[ \quad \text{iff} \quad x \in A \text{ or } (x \in B \text{ and } x \in C) \]
\[ \quad \text{iff} \quad (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \]
\[ \quad \text{iff} \quad x \in A \cup B \text{ and } x \in A \cup C \]
\[ \quad \text{iff} \quad x \in (A \cup B) \cap (A \cup C). \]

Consequently, \(A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\). \qed

4. Let \(A\) and \(B\) be subsets of the sample space \(C\). The \textit{set difference} \(A \setminus B\) is defined to be

\[ A \setminus B = \{ x \in A \mid x \notin B \}; \]

thus, \(A \setminus B\) is a subset of \(A\) that contains those elements in \(A\) which are not in \(B\).

Prove that
(a) \( A \setminus B = A \cap B^c \),

(b) \( B \setminus (A \cap B) = A^c \cap B \)

**Solution:**

(a) *Proof:*

\[
x \in A \setminus B \iff x \in A \text{ and } x \notin B \\
\iff x \in A \text{ and } x \in B^c \\
\iff x \in A \cap B^c.
\]

Consequently, \( A \setminus B = A \cap B^c \). □

(b) *Proof:*

Using the result from part (a), De Morgan’s laws and the distributive property, we get

\[
B \setminus (A \cup B) = B \cap (A \cup B)^c = B \cap (A^c \cup B^c) = (B \cap A^c) \cup (B \cap B^c) = (B \cap A^c) \cup \emptyset = B \cap A^c,
\]

where we have also used the result proved in problem (1)(b). □

5. **Exercise 1 on page 12 in the text**

Suppose that \( A \subseteq B \). Prove that \( B^c \subseteq A^c \).

*Proof:*

If \( x \in B^c \), then \( x \notin B \). It then follows that \( x \notin A \), since \( A \) is a subset of \( B \). Thus, \( x \in A^c \).

We have therefore shown that

\[
x \in B^c \Rightarrow x \in A^c;
\]

in other words, \( B^c \subseteq A^c \). □